LETTER FROM THE EDITOR

We start off 2021 with a fascinating lead article by Michael Gaul and Fred Kuczmarski. They open by racing particles along the sides of elliptical tracks to see which one comes to equilibrium the fastest. This leads to a discussion on the classical problem of proving that an inverted cycloid is a tautochrone, first solved by Christiaan Huygens in 1659.

Those of you who ever find yourselves teaching abstract algebra will want to have a look at the article by Chenggong Du and Daniel R. Shifflet. They take a visual approach to understanding integral domains and their principal ideals. Students often find these objects to be hopelessly abstract, but a bit of geometry can help make them accessible. If you prefer your algebra more linear than abstract, you can have a look at the offering from Loucas Chrysafi and Carlos Marques. They have some novel things to say on the venerable topic of using matrix methods to evaluate sums of integral powers.

Jack E. Graver revisits the problem of enumerating the rational numbers. The usual approach involves traversing the diagonals of a rectangular array. This method scores high marks for accessibility—we routinely show it to students encountering this material for the first time—but it suffers from the defect that it is difficult to identify efficiently where in the list a given number appears. Other known approaches rely on prime factorizations to answer such questions, but since factorization is itself a difficult problem this is not always so helpful. Graver describes a better approach, and along the way introduces readers to the Calkin-Wilf tree.

Pamela Gorkin and Andrew Shaffer discuss a mathematical magic trick: Pick a noncentral point within a disk. Then fold the disk so that the circumference falls precisely on that point. If we do that folding in all possible ways, the creases are the tangent lines to an ellipse with one focus at the given point and the other at the center of the circle. Since this is both awesome and mathematical, it is a safe bet that Martin Gardner wrote about it. Inspired by Gardner, Gorkin and Shaffer discuss a modification of his method that produces a triangle that is inscribed within the circle and circumscribed around the ellipse. Their methods are deeply cool, and I recommend having a look at their paper.

The shorter items will make you smile. Nicholas P. Taliceo and Julian Fleron discuss a sequence of numbers that arises from a problem in ecology. The sequence is interesting because it *seems* to have a connection to the prime number theorem. Bailey Fleugel, Dominic Hatch, and Michael Maltenfort take a clever approach to determining the parity of a given permutation. Finally, we have an interview with John Edmark, a brilliant mathematical artist based at Stanford University. As always, we round out the issue with Problems, Reviews, and Proofs Without Words.

Jason Rosenhouse, Editor

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ARTICLES

Not-So-Simple Pendulums

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Figure 1 shows the starting positions of two point masses on a pair of semi-elliptical tracks. Each track hangs vertically in a uniform gravitational field, and the masses are released simultaneously from rest, sliding without friction in a race to the equilibrium position (E).

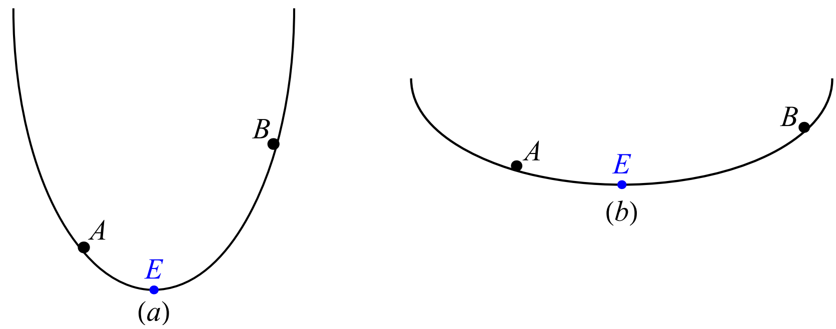

Figure 1 Races along the bottom halves of the elliptical tracks $x^2 + y^2/4 = 1$ (a) and $x^2/4 + y^2 = 1$ (b).

One of these two tracks is *fast* and the mass (B) giving up the head start wins. The other is *slow* and B loses. What does your intuition tell you about which is which?

Preliminaries

Most tracks are neither globally fast nor slow; for some starting positions the mass giving up the head start wins, for other starting positions it loses. The speed of a track is really a local property, and with one notable exception, a track is either *locally* fast or slow. For starting positions sufficiently near equilibrium, the mass giving up the head start always wins (a fast track) or always loses (a slow track).

A track is defined by a smooth function $y = y(s)$ on an interval $s \in (-\epsilon, \epsilon)$, where $y(0) = y'(0) = 0$ and $y''(0) > 0$. The function $y(s)$ gives the height above equilibrium (E) in terms of the distance along the track to E; it is just the y-component of an arclength parameterization. The condition that $y''(0) > 0$ ensures that a track is concave up near equilibrium.

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We use conservation of mechanical energy to determine the race time for a mass sliding along a frictionless track. The mechanical energy of a unit mass in a field with constant acceleration g is the sum $E = v^2/2 + gh$ of its kinetic and potential energies. Here v is the speed of the mass and h is its height. To see that the energy is conserved for a mass dropped from rest and falling only under the influence of gravity, differentiate E with respect to time, giving

$$\frac{dE}{dt} = v\left(\frac{dv}{dt}\right) + g\left(\frac{dh}{dt}\right) = vg - gv = 0.$$

The mass therefore has speed $v = \sqrt{2g\Delta h}$ after falling through a height Δh. Similarly, a mass released from rest at height $y(a)$ on the track $y = y(s)$ has speed

$$v(s) = \sqrt{2g(y(a) - y(s))}$$

at height $y(s)$. The mass traverses the differential arclength ds in the time $ds/v(s)$. For $a > 0$ the function

$$T = \tau(a) = \int_0^a \frac{ds}{v(s)} = \frac{1}{\sqrt{2g}} \int_0^a \frac{ds}{\sqrt{y(a) - y(s)}} \tag{1}$$

gives the travel time to equilibrium.

For a symmetric track, the *time integral* (1) measures the quarter-period of oscillation. But our tracks might not be symmetric and we consider only races between masses released from arclength positions $a > 0$. A track is *slow* (or *fast*) if the *time function* $\tau(s)$ increases (or decreases) with s in a neighborhood $s \in (0, \delta)$. Our problem is to find algebraic conditions on the *height function* $y(s)$, and equivalent geometric conditions on the track, that determine the speed.

We should point out that a simple pendulum, which we typically picture as a mass swinging on the end of a thread, is physically equivalent to a circular track. The force of the track on a sliding mass, directed "inward" and normal to the track because there is no friction, plays the role of the tension in the thread of the swinging pendulum. But we need not think about forces in what follows. Their role is captured in the time integral.

While the time integral played a prominent role in our investigations, it plays little part here other than to produce the graphs in Figures 5 and 6. The discontinuity in the integrand proves just too troublesome, and to get around this we turn to an idea first introduced by Christiaan Huygens some 350 years ago.

The cycloidal pendulum

It did not take long for clockmakers of the 17th century to realize that Galileo was mistaken; the period of a simple pendulum is *not* independent of its amplitude. The Danish clockmaker Christiaan Huygens was the first to discover, in what he described as "the most fortunate finding which ever befell me," that an inverted cycloid is a true tautochrone (from the Greek *tauto chronos*, equal time); a race to equilibrium between masses released simultaneously from rest always ends in a tie.

Huygens' proof, published in his 1673 masterpiece *Horologium Oscillatorium* (*The Pendulum Clock* [**1**]), relies on converting the oscillatory motion of the cycloidal pendulum to circular motion. Figure 2 shows a modified version of an illustration from the *Horologium*. For a mass released from rest at an arbitrary point B of the cycloid $FBPE$, Huygens draws BD perpendicular to the axis and constructs a *tracking semicircle* $\mathcal{C}$ on diameter ED. For each point P of the cycloid between B and E, he associates a *tracking point* Q of $\mathcal{C}$ with the same height. Huygens proves the cycloid is a tautochrone by showing that as P slides down the cycloid, Q rotates around $\mathcal{C}$ at a *constant rate* that is independent of the release point B.

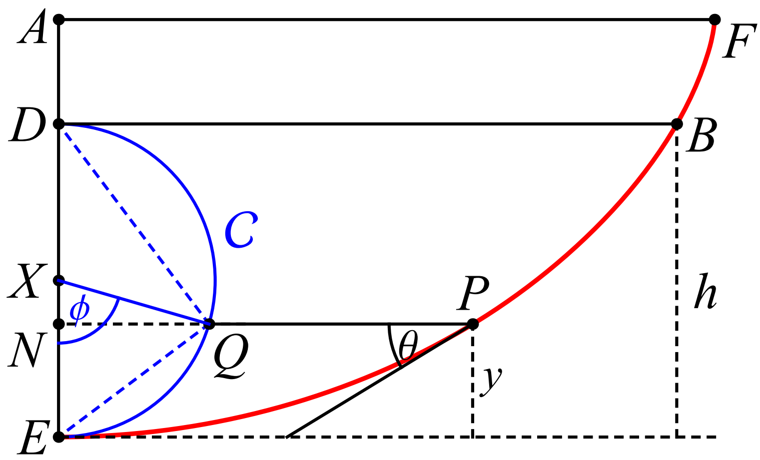

Figure 2 The cycloid and Huygens' tracking semicircle.

For a mass released from a point B of a general track, we construct a tracking semicircle $\mathcal{C}$ on diameter ED just as in Figure 2. Then the tracking point traverses the semicircle in the same time that it takes the mass to slide to equilibrium. So instead of computing the time function (1) directly, we compute (or at least approximate) the time for the tracking point to turn through π radians about the center X of the tracking circle. As a first step in this direction, we determine the rotation rate of the tracking point. The following lemma shows that this rate is a function of the arc length parameter alone and does not depend on the initial height of the mass.

The rotation lemma. *As P slides along the track $y = y(s)$, the tracking point Q rotates about the center of the tracking semicircle at the rate $\Omega(s) = \sqrt{2g/y}\,(dy/ds)$.*

Proof. To avoid unnecessary negative signs, we suppose the mass is launched from equilibrium at time $t = 0$ and comes to rest at B with height $h = 2r$, where r is the radius of $\mathcal{C}$. Let P be a point of the track at height $y = y(s) \le h$ and let θ be the angle between the track and the horizontal at P. Then the mass passes P with speed $v_P = \sqrt{2g(h-y)}$, ascending at the rate

$$\dot{y} = \frac{dy}{dt} = v_P \sin\theta = \sqrt{2g(h-y)}\left(\frac{dy}{ds}\right).$$

Now let $\phi = \angle EXQ$ be the *tracking angle*. Then

$$y = r(1-\cos\phi) \qquad \text{and} \qquad \dot{y} = r\dot{\phi}\sin\phi.$$

Hence,

$$\Omega(s) = \dot{\phi} = \frac{\dot{y}}{r\sin\phi} = \frac{\dot{y}}{QN}.$$

But since triangles ΔDNQ and ΔQNE are similar,

$$\frac{DN}{QN} = \frac{QN}{NE} \qquad \text{and} \qquad QN = \sqrt{NE \cdot DN}.$$

Thus,

$$\Omega(s) = \frac{\dot{y}}{QN} = \frac{\dot{y}}{\sqrt{y(h-y)}} = \sqrt{\frac{2g}{y}}\left(\frac{dy}{ds}\right).$$

■

Now suppose that $\Omega(s) = 2\omega$ for some constant ω. Then since the tracking point turns through π radians on its way to equilibrium and $\Omega(s)$ is independent of the starting height, the travel time $\tau(s) = \pi/(2\omega)$ is also independent of the starting height. Such a track is thus a tautochrone.

Now, in general, the half-rotation rate of the tracking point is

$$\frac{\Omega(s)}{2} = \sqrt{\frac{g}{2y}}\left(\frac{dy}{ds}\right) = \frac{d(\sqrt{2gy})}{ds}. \tag{2}$$

In particular, for the tautochrone with $\Omega(s) = 2\omega$, we have $d\sqrt{2gy} = \omega\, ds$. Using the initial condition $y(0) = 0$ gives $\sqrt{2gy} = \omega s$ and the height function of the tautochrone as

$$y = \left(\frac{\omega^2}{2g}\right)s^2. \tag{3}$$

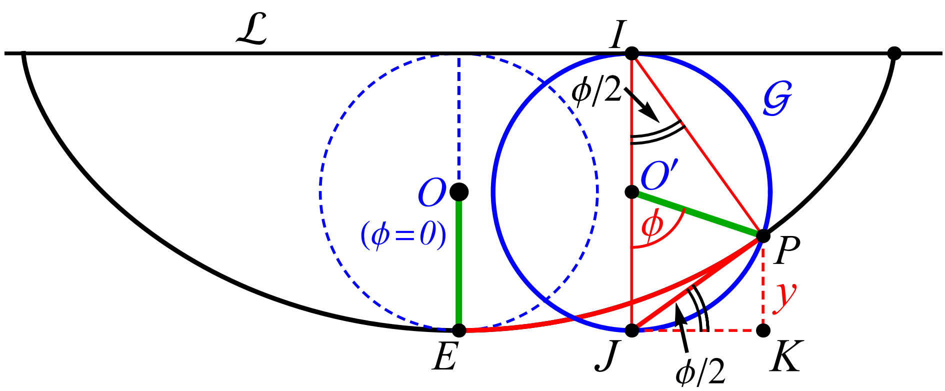

Figure 3 The height function of a cycloid.

To see that (3) defines a cycloid, consider the cycloid of Figure 3, traced by a point P on a circle $\mathcal{G}$ of radius r rolling on a line $\mathcal{L}$. With the tracing point in its initial position at E, mark the vertical radius OE. Now suppose $\mathcal{G}$ rolls to some other position and let ϕ, the angle between the vertical and the marked radius $O'P$, measure the rotation angle of $\mathcal{G}$. Since the differential motion of $\mathcal{G}$ is a rotation through the angle $d\phi$ about its point of contact I with $\mathcal{L}$, P sweeps out the differential arclength

$$ds = IP\, d\phi = 2r\cos(\phi/2)\, d\phi.$$

Hence,

$$s = \int_0^{\phi} 2r\cos(u/2)\, du = 4r\sin(\phi/2)$$

and

$$y = r(1-\cos\phi) = 2r\sin^2(\phi/2) = \frac{s^2}{8r}. \tag{4}$$

Comparing (3) and (4) shows that $\omega = \sqrt{g/4r}$ and that the quarter-period (i.e., the travel time to equilibrium) of the cycloidal pendulum is $\tau = \pi/(2\omega) = \pi\sqrt{r/g}$.

Small oscillations

With the rotation lemma, we can approximate the quarter-period for small oscillations about equilibrium on any track. First note that the discontinuity in $\Omega(s) = \sqrt{2g/y}\,(dy/ds)$ at $s = 0$ is removable. For, since

$$\lim_{s\to 0}\Omega^2(s) = 2g\lim_{s\to 0}\frac{(dy/ds)^2}{y} = 2g\lim_{s\to 0}\frac{2(dy/ds)(d^2y/ds^2)}{dy/ds} = 4gy''(0),$$

it follows that

$$\lim_{s\to 0}\ \Omega(s) = 2\sqrt{gy''(0)} = 2\sqrt{\frac{g}{\rho_0}},$$

where $\rho_0 = 1/y''(0)$ is the radius of curvature of the track at equilibrium. To understand this last equality, recall that the curvature $\kappa(s) = d\theta/ds$ measures the rate at which the track changes direction relative to arclength. Here, as in Figure 2, we take θ to be the track's angle of inclination. Since $\sin\theta = dy/ds$,

$$\kappa(s)\cos\theta = \frac{d^2y}{ds^2}, \tag{5}$$

and

$$\rho(s) = \frac{1}{\kappa(s)} = \frac{\cos\theta}{(d^2y/ds^2)}$$

is the radius of curvature. But at equilibrium, $\theta = 0$, and so $\rho(0) = 1/y''(0)$.

Keeping in mind that the tracking point traverses a semicircle on its way to equilibrium, we get that for $a \sim 0$,

$$\tau(a) \sim \lim_{s\to 0}\tau(s) = \frac{\pi}{\lim_{s\to 0}\ \Omega(s)} = \frac{\pi}{2}\sqrt{\frac{\rho_0}{g}}.$$

In particular, for small amplitudes the quarter-period of a simple pendulum of length ℓ (and constant radius of curvature $\rho_0 = \ell$) is approximately $(\pi/2)\sqrt{\ell/g}$.

Track speed

We can use the rotation lemma to determine the speed of a track by noting that since the discontinuity in $\Omega(s)$ at $s = 0$ is removable, there is a sufficiently small neighborhood $s \in (0, \delta)$ on which $\Omega(s)$ is either strictly monotonic or constant.

The track lemma. *A track is locally fast when $\Omega(s)$ increases for $s \in (0, \delta)$ and locally slow when $\Omega(s)$ decreases for $s \in (0, \delta)$. Otherwise, the track is a tautochrone and an arc of a cycloid.*

Proof. The idea of the proof is to convert a short race between two masses to a race between their tracking points. We then determine the winner by comparing the rotation rates of the tracking points at congruent tracking angles.

Figure 4 shows a race between masses released from B and B' at respective arc length parameters $s = \delta$ and $s = \delta' < \delta$. In the figure, P denotes the position of the mass released from B at some time during the race; Q is the tracking point for P. Now let Q' be the point where the segment EQ intersects the tracking circle for B'. Since ΔEXQ and $\Delta EX'Q'$ are isosceles, we have that

$$\angle XQE \cong \angle XEQ \cong \angle X'Q'E.$$

Hence, Q and Q' have congruent tracking angles $\angle EXQ$ and $\angle EX'Q'$.

Now let P' be the point on the track having Q' as its tracking point. We know that P' is below P because Q' is below Q. So if $\Omega(s)$ increases for $s \in (0, \delta)$, then Q rotates faster than Q'. Since this is true at *each* tracking angle, Q beats Q' in the race to equilibrium and the track is fast. Similarly, the track is slow if $\Omega(s)$ decreases.

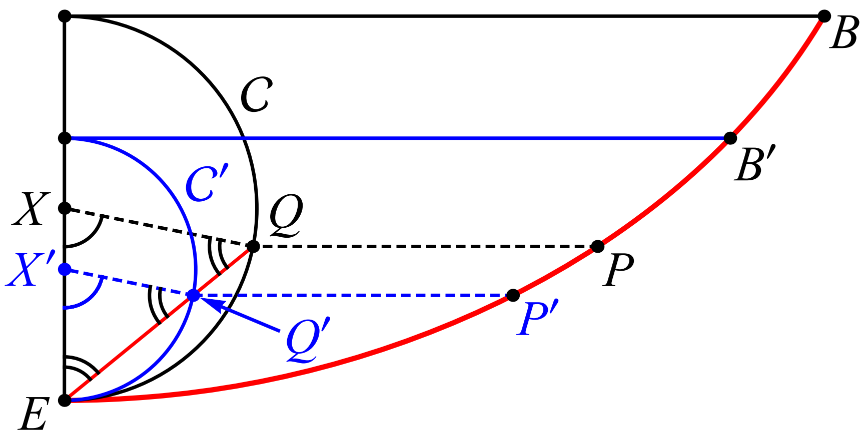

Figure 4 Determining the speed of a track.

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We saw earlier that the tracking point for the cycloid $y = (\omega^2/2g)s^2$ (see (3)) rotates at the constant rate 2ω. This suggests that the behavior of the rotation rate $\Omega(s)$ near $s = 0$, and hence the speed of the track $y = y(s)$, is determined by the first non-zero derivative $y^{(n)}(0)$ of order $n \geq 3$. So for integers $n \geq 3$ we define an *n-track* to have a height function of the form

$$y(s) = \frac{s^2}{2\rho_0} + b_n s^n + o(s^n), \tag{6}$$

where $b_n = y^{(n)}(0)/n! \neq 0$. Here the "little o" notation means that $e(s) = y(s) - s^2/(2\rho_0) - b_n s^n$ approaches 0 *faster* than s^n in the sense that $\lim_{s\to 0} e(s)/s^n = 0$.

To determine the speed of the n-track (6), we first approximate $\sqrt{y(s)}$. From Newton's binomial theorem, $\sqrt{1+x} = 1 + x/2 + o(x)$. This and some algebra gives

$$\sqrt{y(s)} = \sqrt{1/(2\rho_0)}\left(s + \rho_0 b_n s^{n-1}\right) + o(s^{n-1}). \tag{7}$$

As a check, just square the right side of (7) and use the fact that $s\, o(s^{n-1}) = o(s^n)$. This is enough to determine the track's speed.

The speed of a track. *The n-track $y = y(s)$ is locally fast if $y^{(n)}(0) > 0$ and locally slow if $y^{(n)}(0) < 0$.*

Proof. Let $\omega = \sqrt{g/\rho_0}$ and differentiate (7) with respect to s. Using (2), this gives the *half-rotation rate* of the tracking point as

$$\Omega(s)/2 = d(\sqrt{2gy})/ds = \omega\left(1 + \rho_0 b_n (n-1)s^{n-2}\right) + o(s^{n-2}). \tag{8}$$

When $b_n > 0$, the rotation rate is an increasing function of s and the track is fast by the track lemma. Similarly, the track is slow when $b_n < 0$.

■

In particular, the simple pendulum of length ℓ with height function

$$y = \ell\left(1 - \cos\left(\frac{s}{\ell}\right)\right) = \frac{s^2}{2\ell} - \frac{s^4}{24\ell^3} + o(s^4), \quad 0 \le s \le \pi\ell,$$

is locally slow. Furthermore, since $y = 2\ell \sin^2(s/2\ell)$,

$$\frac{\Omega(s)}{2} = \frac{d(\sqrt{2gy})}{ds} = \sqrt{\frac{g}{\ell}}\left(\cos\left(\frac{s}{2\ell}\right)\right). \tag{9}$$

Since $\Omega(s)$ decreases for $s \in (0, \pi\ell)$, the track lemma implies that the simple pendulum is *globally* slow.

Tautochrone-like tracks

Figure 5 shows the graph of the time function $T = \tau(s)$ (see (1)) for the simple pendulum. It suggests that short races to equilibrium nearly end in a tie and explains why Galileo mistakenly believed the simple pendulum to be a tautochrone. We call such a track, where $\tau'(0) = 0$, *tautochrone-like*.

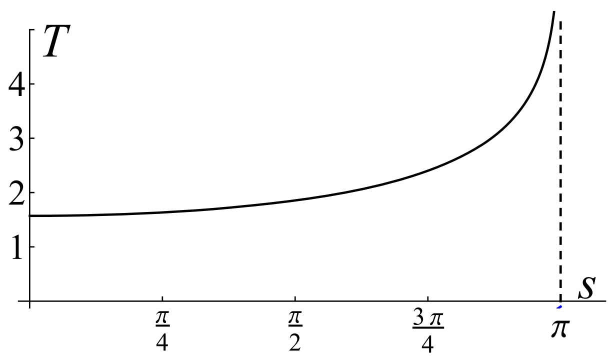

Figure 5 The time function $T = \tau(s)$ for the simple pendulum $y = 1 - \cos s$, with $g = 1$.

Even tracks, where $y(s)$ is an even function, are necessarily tautochrone-like. But tautochrone-like tracks are not necessarily even. Figure 6 suggests the track $y = s^2/2 + 10s^5$ is tautochrone-like, but that the track $y = s^2/2 + s^3 + 10s^4$ is not. Experiments with Mathematica convinced us that the n-track $y = y(s)$ is tautochrone-like when $n \ge 4$, or equivalently when $y^{(3)}(0) = 0$.

To prove this claim, we abandon the time function in its form (1) and instead approximate the time for the tracking point to turn through π radians on its way to equilibrium. The rotation lemma gives the rotation rate of the tracking point in terms of the track's arc length parameter, but to approximate the time function we need to express this rate in terms of the tracking angle. This entails expressing, or at least approximating, the arclength parameter s in terms of the half-tracking angle $\phi^* = \phi/2$.

As in Figure 2, let a mass be released from height $h = y(a)$ on an n-track. Then for $0 \le s \le a$, the height of the mass at arc length parameter s is

$$y(s) = \frac{y(a)(1 - \cos\phi)}{2} = y(a)\sin^2\phi^*. \tag{10}$$

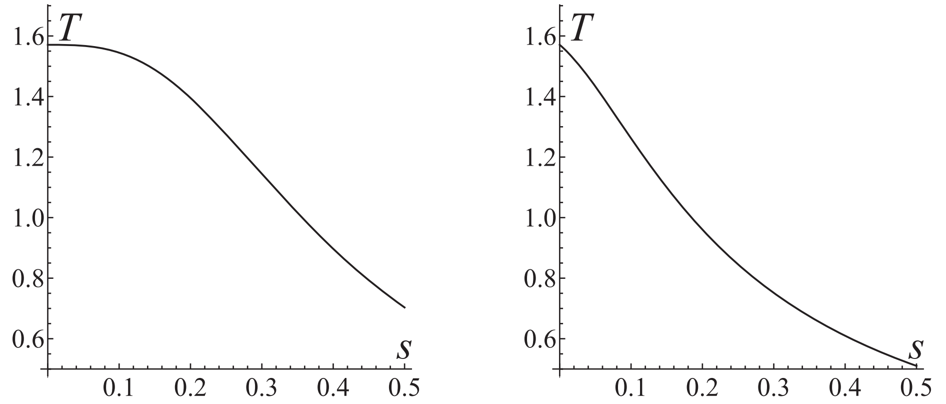

Figure 6 Time functions for the tracks $y = s^2/2 + 10s^5$ (left) and $y = s^2/2 + s^3 + 10s^4$ (right), with $g = 1$.

Before tackling the general case, we sketch our strategy for the simple pendulum, where we can solve this equation explicitly for s. Using the height function

$$y(s) = \ell\left(1 - \cos\left(\frac{s}{\ell}\right)\right) = 2\ell \sin^2\left(\frac{s}{2\ell}\right)$$

of the simple pendulum in (10) gives

$$\sin^2\left(\frac{s}{2\ell}\right) = \sin^2\left(\frac{a}{2\ell}\right)\sin^2\phi^*.$$

So from (9) the half-rotation rate Ω^* of the tracking point is

$$\Omega^* = \left(\sqrt{\frac{g}{\ell}}\right)\cos\left(\frac{s}{2\ell}\right) = \left(\sqrt{\frac{g}{\ell}}\right)\sqrt{1 - \sin^2\left(\frac{a}{2\ell}\right)\sin^2\phi^*}.$$

Since the half-angle decreases from $\phi^* = \pi/2$ to $\phi^* = 0$ as the pendulum falls to equilibrium, the simple pendulum has quarter-period

$$\tau(a) = \int_0^{\pi/2}\frac{d\phi^*}{\Omega^*} = \sqrt{\frac{\ell}{g}}\int_0^{\pi/2}\frac{d\phi^*}{\sqrt{1 - \sin^2(a/2\ell)\sin^2\phi^*}}.$$

Finally, using the approximations $(1-x)^{-1/2} \sim 1 + x/2$ and $\sin\theta \sim \theta$ gives

$$\tau(a) \sim \sqrt{\frac{\ell}{g}}\left(\frac{\pi}{2} + \frac{a^2}{8\ell^2}\int_0^{\pi/2}\sin^2\phi^*\,d\phi^*\right) = \frac{\pi}{2}\sqrt{\frac{\ell}{g}}\left(1 + \frac{a^2}{16\ell^2}\right).$$

The general case works similarly, except that we use (10) to approximate s in terms of a. To bound the error, fix the half-angle $\phi^* = \phi/2$, so that (10) defines s implicitly as a function of a. Now ignoring all but the quadratic term, write the n-track (6) as $y(s) = s^2/(2\rho_0) + o(s^{n-1})$. Then (10) becomes

$$s^2 + o(s^{n-1}) = a^2\sin^2\phi^* + o(a^{n-1}).$$

Since $0 \le s \le a$,

$$s^2 = a^2\sin^2\phi^* + o(a^{n-1}).$$

Keeping in mind that $n \geq 3$ gives

$$\lim_{a\to 0}(s^2/a^2) = \sin^2\phi^*.$$

Thus, $\lim_{a\to 0}(s/a) = \sin\phi^*$ and $s = a\sin\phi^* + o(a)$.

Now with s as a function of a, we can rewrite the half-rotation rate (8) as

$$\Omega^* = \omega\left(1 + \rho_0 b_n(n-1)s^{n-2}\right) + o(a^{n-2}).$$

Then since $(x + o(x))^k = x^k(1 + o(1))^k = x^k(1 + o(1)) = x^k + o(x^k)$,

$$s^{n-2} = (a\sin\phi^* + o(a))^{n-2} = a^{n-2}\sin^{n-2}\phi^* + o(a^{n-2}),$$

and

$$\Omega^* = \omega\left(1 + \rho_0 b_n(n-1)a^{n-2}\sin^{n-2}\phi^*\right) + o(a^{n-2}).$$

Finally, using the approximation $(1+x)^{-1} = 1 - x + o(x)$ for $1/\Omega^*$ and recalling that $\omega = \sqrt{g/\rho_0}$ gives

$$\begin{aligned}\tau(a) &= \int_0^{\pi/2}\frac{d\phi^*}{\Omega^*} = \sqrt{\frac{\rho_0}{g}}\int_0^{\pi/2}\left(1 - \rho_0 b_n(n-1)a^{n-2}\sin^{n-2}\phi^*\right)\,d\phi^* + o(a^{n-2})\\ &= \sqrt{\frac{\rho_0}{g}}\left(\frac{\pi}{2} - \rho_0 b_n(n-1)a^{n-2}\int_0^{\pi/2}\sin^{n-2}\phi^*\,d\phi^*\right) + o(a^{n-2}).\end{aligned}$$

Substituting $b_n = y^{(n)}(0)/n!$, we have the following lemma.

The timing lemma. *The first non-zero derivative $\tau^{(k)}(0)$ of the time function for the n-track $y = y(s)$ is*

$$\tau^{(n-2)}(0) = -\frac{y^{(n)}(0)}{n}\sqrt{\frac{\rho_0^3}{g}}\int_0^{\pi/2}\sin^{n-2}\phi\,d\phi.$$

In particular, since

$$\tau'(0) = -\frac{y^{(3)}(0)}{3}\sqrt{\frac{\rho_0^3}{g}}, \tag{11}$$

an n-track is tautochrone-like if and only if $n \geq 4$. For a geometric interpretation of this condition, differentiate the relation $y''(s) = \kappa(s)\cos\theta$ (see (5)) with respect to s, giving

$$y^{(3)}(s) = \kappa'(s)\cos\theta - (\kappa(s))^2\sin\theta. \tag{12}$$

Hence, $y^{(3)}(0) = \kappa'(0)$ and *a track is tautochrone-like when equilibrium is a critical point of the curvature function*. For a more vivid interpretation, we return to the *Horologium Oscillatorium*.

Pendulums and evolutes

Even before he realized that the cycloid was a tautochrone, Huygens had attached curved plates to the fulcrum of his pendulum clock to compensate for the slowness of the simple pendulum. The thread of the pendulum wrapped around the plates, lifting

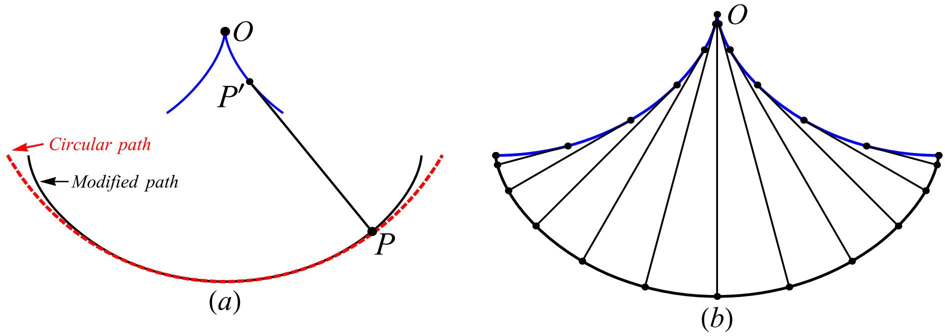

Figure 7 Adjusting for the slowness of the simple pendulum (a); the cycloidal pendulum (b).

the bob above its circular path and decreasing the period for larger amplitudes. (See Figure 7(a), where the thread is tangent to the plate at P'.)

Huygens later made the discovery that bending the plates in the form of a cycloid forces the pendulum bob to follow the path of a congruent cycloid and act like a true tautochrone (Figure 7(b)). More generally, he showed how to force a pendulum bob to follow any smooth path $\mathcal{C}$ by bending the plates in the shape of the *evolvent* (or unwound) curve. Today we call this curve the *evolute*. Its points are the centers of curvature of $\mathcal{C}$.

This suggests we supplement the image of a mass sliding along a track with that of a pendulum whose thread winds (or unwinds) around the track's evolute. The motion of the pendulum, and hence that of the sliding mass, is determined by the geometry of the track's evolute and what we call the pendulum's *length* $\rho_0 = OE$ (see Figure 8). In keeping with a simple pendulum, where the evolute collapses to a point, we call O the *fulcrum*. It is the track's center of curvature at equilibrium.

To exploit this perspective, we need to relate the geometry of the track to the geometry of its evolute. The key is the relation

$$\rho_1 = \left| \frac{d\rho}{d\theta} \right|, \tag{13}$$

that expresses the radius of curvature $\rho_1 = \rho_1(s)$ of the evolute in terms of the radius of curvature $\rho = \rho(s)$ of the track. We sketch a proof of this in the appendix, but for now note that (13) is dimensionally correct; $d\rho/d\theta$ is indeed a length.

An immediate consequence of (13) is that *a track is tautochrone-like when the radius of curvature of its evolute vanishes at the fulcrum.* For since

$$\frac{d\kappa}{ds} = \frac{d(1/\rho)}{ds} = -\frac{d\rho}{ds} \cdot \frac{1}{\rho^2} = -\frac{d\rho}{d\theta} \cdot \frac{d\theta}{ds} \cdot \frac{1}{\rho^2} = -\frac{d\rho}{d\theta} \cdot \frac{1}{\rho^3}, \tag{14}$$

the derivatives $\kappa'(s)$ and $\dot{\rho}(s) = d\rho/d\theta$ vanish simultaneously. Thus, a track is tautochrone-like when $\dot{\rho}(0) = 0$ and hence by (13) when $\rho_1(0) = 0$.

Near equilibrium, an n-track behaves much like a pendulum whose thread wraps around the osculating circle of the track's evolute at the fulcrum. When $n \geq 4$, like the tracks in Figure 9, the osculating circle collapses to a point. This point-like behavior of the evolute is enough to preserve the tautochrone-like property of the simple pendulum. But the osculating circle of a 3-track does not collapse, and this destroys the tautochrone-like property.

Sketching the evolute of a 3-track, or even just the evolute's osculating circle at the fulcrum, reveals the track's speed. As in Figure 8(a), the track is fast when $y^{(3)}(0) =$

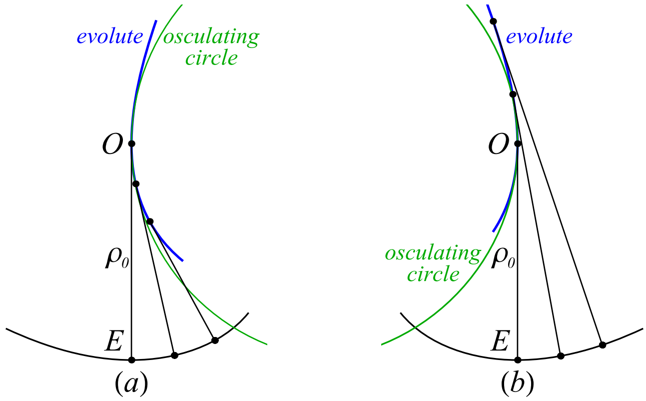

Figure 8 A pendulum wrapping around the evolute (blue) of the fast track $y = s^2/2 + s^3/6$ (a) and unwrapping from the evolute of the slow track $y = s^2/2 - s^3/6$ (b).

$\kappa'(0) > 0$ and the radius of curvature (that is, the length of the unwound part of the string) *decreases* as the pendulum is displaced from equilibrium. Then the evolute lifts the pendulum high enough above the arc of a simple pendulum to convert it into a fast track. But when $y^{(3)}(0) = \kappa'(0) < 0$, the thread unwraps from the evolute and the track is slow (Figure 8(b)).

Furthermore, from (14) and then (13)

$$y^{(3)}(0) = \kappa'(0) = \frac{-\dot{\rho}(0)}{\rho_0^3} = \frac{-\operatorname{sgn}(\dot{\rho}(0))\rho_1(0)}{\rho_0^3},$$

and we can rewrite (11) as

$$\tau'(0) = -\frac{y^{(3)}(0)}{3}\sqrt{\frac{\rho_0^3}{g}} = \operatorname{sgn}(\dot{\rho}(0))\frac{\rho_1(0)}{3\sqrt{\rho_0^3 g}}.$$

This shows how the evolute and the thread length affect a pendulum's motion; $\tau'(0)$ is directly proportional to the radius of curvature of the evolute at the fulcrum and inversely proportional to $\rho_0^{3/2}$.

The effect of the evolute on the motion of a tautochrone-like track where the radius of curvature vanishes at the fulcrum, like those of the elliptical tracks from the introduction shown again in Figure 9, is more subtle. The speeds of these 4-tracks depend upon the sign of $y^{(4)}(0)$. Differentiating (12) with respect to s gives

$$y^{(4)}(s) = \kappa''(s)\cos\theta - (\kappa'(s))^2\sin\theta - 2\kappa(s)\kappa'(s)\sin\theta - (\kappa(s))^3\cos\theta$$

and $y^{(4)}(0) = \kappa''(0) - (\kappa(0))^3$. So when equilibrium is a local maximum of the curvature function like it is for the track in Figure 9(a), the track is slow. For then $\kappa''(0) \le 0$ and $y^{(4)}(0) < 0$. Just like the slow 3-track in Figure 8(b), the thread of the pendulum unwraps from the evolute as the bob is displaced from equilibrium.

But when equilibrium is a local minimum of the curvature function, as it is in Figure 9(b), we need to determine whether the evolute lifts the bob high enough above a circular arc to convert the simple pendulum into a fast track. To relate the derivative $y^{(4)}(0)$ to the geometry of the evolute, note first that since the radius of curvature decreases away from equilibrium, we can write (13) as $\rho_1 = -d\rho/d\theta$. Then (14)

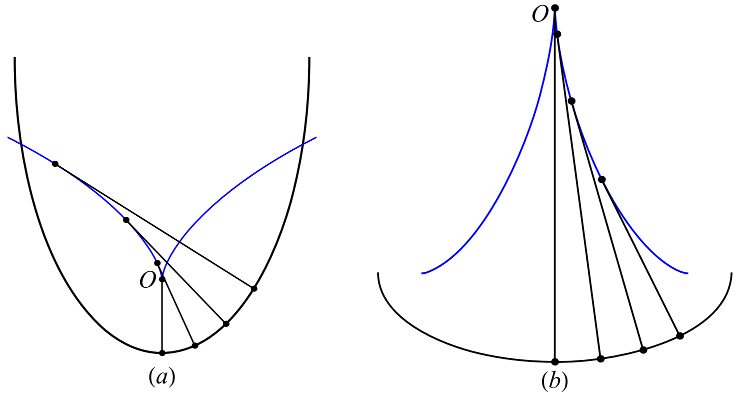

Figure 9 The elliptical tracks of Figure 1 with their evolutes. Track (a) is slow.

becomes $d\kappa/ds = \rho_1/\rho^3$. Since $\dot\rho(0) = 0$,

$$\kappa''(0) = \frac{d}{d\theta}\left(\frac{\rho_1}{\rho^3}\right) \cdot \frac{d\theta}{ds}\bigg|_{s=0} = \left(\frac{\rho^3\dot\rho_1 - 3\rho^2\dot\rho\rho_1}{\rho^7}\right)\bigg|_{s=0} = \frac{\dot\rho_1(0)}{\rho_0^4}.$$

Thus,

$$y^{(4)}(0) = \kappa''(0) - (\kappa(0))^3 = \frac{\dot\rho_1(0) - \rho_0}{\rho_0^4} = \frac{\rho_2(0) - \rho_0}{\rho_0^4},$$

where $\rho_2(0) = |\dot\rho_1(0)| = \dot\rho_1(0)$* is the radius of curvature of the track's *second evolute* (the evolute of the evolute) at $s = 0$. This radius measures how quickly the evolute deviates from its point-like behavior near the fulcrum.

So when the thread wraps around the evolute of a tautochrone-like track, a sufficiently long pendulum ($\rho_0 > \rho_2(0)$) dampens the effect of the evolute enough to preserve the slowness of the simple pendulum. Otherwise, when $\rho_0 < \rho_2(0)$, the track is fast. These conditions give a way to visualize the speed of a 4-track by plotting the track and its first two evolutes. Figure 10 shows the elliptical track of Figure 9(b) with its evolutes. The "flatness" of the second evolute at the fulcrum reveals just how rapidly the first evolute deviates from its point-like behavior and suggests that $\rho_0 < \rho_2(0)$. A

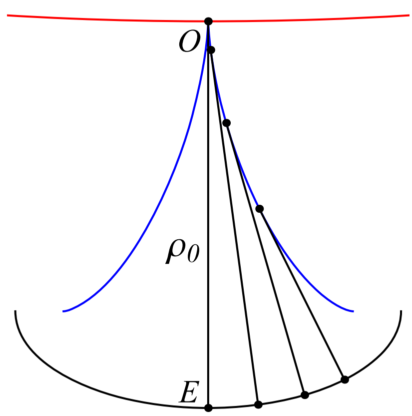

Figure 10 The elliptical 4-track $f(x) = -\sqrt{1 - x^2/4}$ of Figure 1(b) with its first evolute (blue) and second evolute (red). The track is fast since $\rho_0 < \rho_2(0)$.

*The first equality follows from (13), the second from the fact ρ_1 is increasing since $\rho_1(0) = 0$.

little help from Mathematica confirms this ($\rho_0 = 4$ and $\rho_2(0) = 36$) and the track is fast.

Appendix

We prove here the relation $\rho_1 = |d\rho/d\theta|$ (see (13)) that expresses the radius of curvature of a track's evolute in terms of the track's radius of curvature. There are two key points.

(a) Since the thread wraps around or unwraps from the evolute, the arc length of the evolute measured from the fulcrum is $s_1 = |\rho(s) - \rho_0|$. So in Figure 11, for example, arc $OP' = |P'P - OE|$. This implies the differential relation $ds_1 = |d\rho|$.

(b) Because the differential motion of the thread $P'P$ is a rotation about P', segment $P'P$ is normal to the track. So the angle between $P'P$ and the vertical is equal to the angle θ between the track and the horizontal at P.

It follows that the radius of curvature of the evolute is $\rho_1 = ds_1/d\theta = |d\rho/d\theta|$.

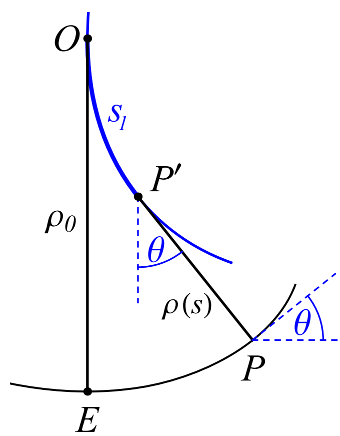

Figure 11 A track and its evolute.

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[1] Blackwell, R. J. (1986). *Christiaan Huygens' The Pendulum Clock or Geometrical Demonstration Concerning the Motion of Pendula as Applied to Clocks*, Ames: The Iowa State University Press.

Summary. Borrowing ideas from Christiaan Huygens' *Horologium Oscillatorium* (*The Pendulum Clock*), we analyze the motion of a general pendulum with small amplitudes.

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Geometry and Principal Ideals

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We explore two questions about integral domains of the form

$$\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} | a, b \in \mathbb{Z}\},$$

where $d \neq 1$ is *square-free*, meaning not divisible by the square of a prime.

1. Is there a reasonable geometric interpretation of $\mathbb{Z}[\sqrt{d}]$ within the Cartesian plane that offers insight into its properties?
2. Can we use this new insight to gain a deeper understanding of the integral domain?

The answer to both of these questions is "Yes!" By visualizing $\mathbb{Z}[\sqrt{d}]$ as a concrete set of points in the plane instead of as an abstract ring, we can draw on the geometric properties of this space to help us further explore characteristics like units and principal ideals. From this new viewpoint a very interesting connection to the norm of an element will be realized.

Representing $\mathbb{Z}[\sqrt{d}]$ geometrically

We can identify elements of the form $a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ with the lattice points (a, b) in the Cartesian plane. The integral domain then appears as the discrete space shown in Figure 1.

This seems like a good start. We have a geometric interpretation of the integral domain $\mathbb{Z}[\sqrt{d}]$ that is similar to structures with which we are familiar. Of course, the true test is to see if other interesting features of $\mathbb{Z}[\sqrt{d}]$ are revealed by this geometric perspective.

Representing units We know that every nonzero element of $\mathbb{C}$ is a unit (that is, every nonzero complex number has a multiplicative inverse). The set of integers, on the other hand, contains only two elements with inverses: 1 and -1. The geometric interpretation of units in these cases is consequentially anticlimactic. But the units in $\mathbb{Z}[\sqrt{d}]$ are more interesting. If $d > 0$, for example, then $\mathbb{Z}[\sqrt{d}]$ has an infinite number of units [**1**, p. 230]. So what do they look like in our geometry?

Consider that an element $a + b\sqrt{d}$ of $\mathbb{Z}\sqrt{d}$ is a unit if and only if there is a solution $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ to the equation

$$(a + b\sqrt{d})(x + y\sqrt{d}) = 1.$$

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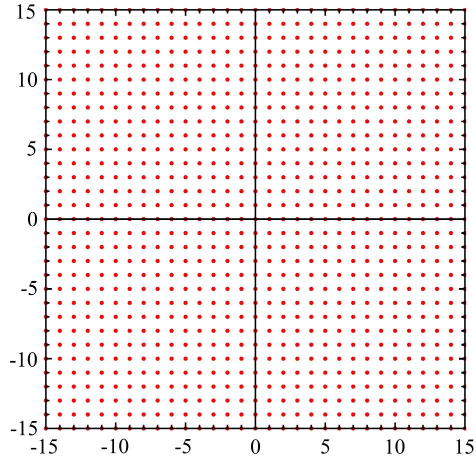

Figure 1 The points represent a geometric interpretation of $\mathbb{Z}[\sqrt{d}]$.

A little high school algebra reveals

$$\frac{a}{a^2 - db^2} - \frac{b}{a^2 - db^2}\sqrt{d}$$

to be the only possibility for $x + y\sqrt{d}$. However, this element is only in the ring $\mathbb{Z}[\sqrt{d}]$ itself when the denominator $a^2 - db^2$ equals 1 or -1. Thus, finding units in $\mathbb{Z}[\sqrt{d}]$ reduces to finding ordered pairs (a, b) satisfying the equation $|a^2 - db^2| = 1$.

Had we been working in $\mathbb{R} \times \mathbb{R}$, the graph of solutions to this equation would be an ellipse (if $d < 0$) or a pair of hyperbolas (if $d > 0$). But in our geometry of $\mathbb{Z}[\sqrt{d}]$, the graph of solutions corresponds only to the points along these curves with integer coordinates. If our graph in $\mathbb{R} \times \mathbb{R}$ would have been an ellipse, then the units in $\mathbb{Z}[\sqrt{d}]$ correspond to the points $(\pm 1, 0)$ and $(0, \pm 1)$ if $d = -1$, or just $(\pm 1, 0)$ if $d < -1$. But if our graph in $\mathbb{R} \times \mathbb{R}$ would have been a pair of hyperbolas, then the units in $\mathbb{Z}[\sqrt{d}]$ follow a pattern like the points highlighted in Figure 2.

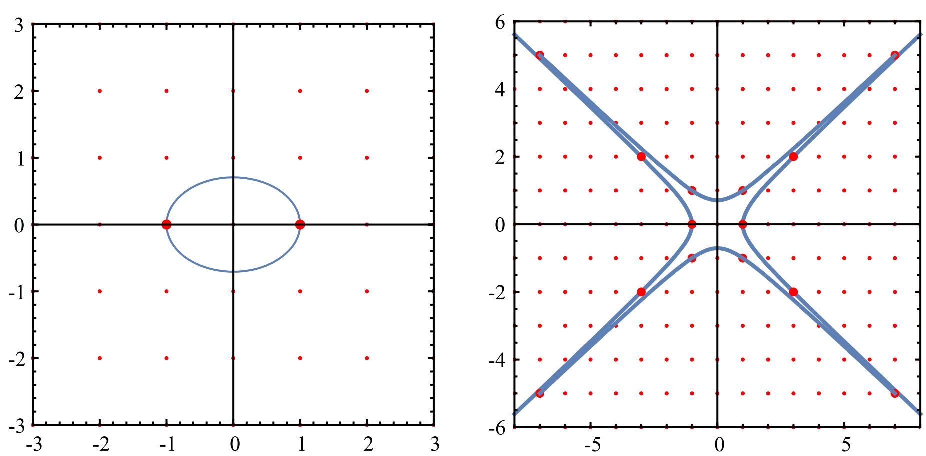

Figure 2 The highlighted points are units of $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{2}]$ as they represent integer solutions to $|a^2 + 2b^2| = 1$ and $|a^2 - 2b^2| = 1$, respectively.

The expression $|a^2 - db^2|$ is called the *norm* of an element and is denoted by $N(a + b\sqrt{d})$. In other branches of mathematics, such as analysis, the norm of an element tends to correspond to a measure of distance from 0. But notice that our units, each with a norm of 1, can be located arbitrarily far from the origin. Does this mean the norm fails to have a geometric interpretation in our particular ring? Not at all! We simply have to dig a little deeper.

Representing ideals Consider a specific $a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ where the coefficients a and b are not both 0. This one element generates the *principal ideal*

$$\langle a + b\sqrt{d}\rangle = \{(a + b\sqrt{d})(x + y\sqrt{d}) \mid x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]\}.$$

Now return to Figure 1. If our integral domain $\mathbb{Z}[\sqrt{d}]$ is represented geometrically by this grid-like discrete set of points, what would a principal ideal in this domain look like? It would obviously have to be a subset of elements that includes the representation of our generator (a, b). Then, since ideals are subrings and subrings are closed under addition, it would have to include all integer multiples of this point. This would give us a discrete "line" of points

$$\ldots, (-3a, -3b),\ (-2a, -2b),\ (-a, -b),$$
$$(0, 0),\ (a, b),\ (2a, 2b),\ (3a, 3b), \ldots$$

passing through the origin (see Figure 3).

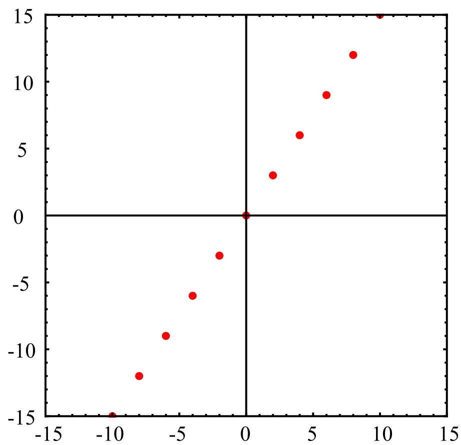

Figure 3 Integer multiples of the ordered pair $(2, 3)$ must be elements of $\langle 2 + 3\sqrt{-1}\rangle$ in $\mathbb{Z}[\sqrt{-1}]$.

But ideals are also closed under multiplication by *any* element of the ring. So what happens when we take the points on this "line" and start multiplying them by all the other points found in Figure 1 (thinking of all points as elements of $\mathbb{Z}[\sqrt{d}]$)? It may seem like we would generate the entire discrete plane. But this only happens if the generator of our ideal is a unit. In other cases we get a much more interesting subset of points. For example, the principal ideal $\langle 2 + 3\sqrt{-1}\rangle$ in $\mathbb{Z}[\sqrt{-1}]$ contains exactly the elements pictured in Figure 4.

At this point we should feel like we are on the right path. All of the algebraic rigor of a principal ideal ends up translating into a very intuitive geometric representation. A few more examples are provided in Figure 5, for clarity.

While illuminating in their own right, considering these pictures side by side brings about new questions. In particular, what makes one ideal look more "dense" than another? That is, why does it contain more points in the same amount of area? The answer brings us back to the norm.

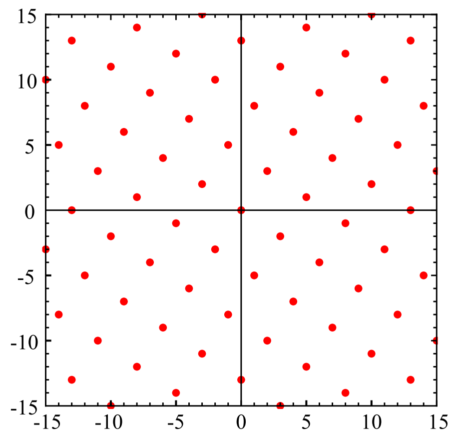

Figure 4 The ideal $\langle 2+3\sqrt{-1}\rangle$ in $\mathbb{Z}[\sqrt{-1}]$ has a very symmetrical geometric interpretation.

The density of $\mathbb{Z}[\sqrt{d}]$

Intuitively, we might think of the density of an ideal I in a ring R as the proportion of elements in R that are in I. But what does this mean when I and/or R are infinite? It means we must derive a more precise definition of density.

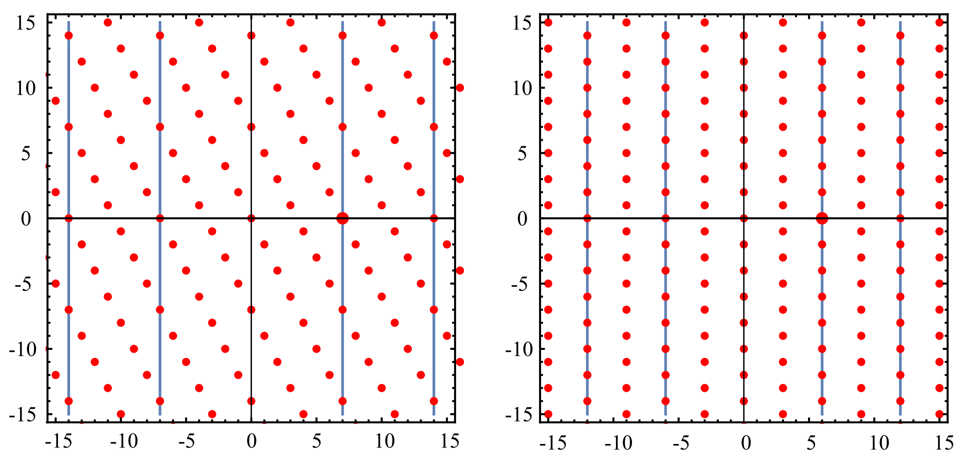

Figure 5 The ideals $\langle 3+1\sqrt{2}\rangle$ of $\mathbb{Z}[\sqrt{2}]$ and $\langle 3+1\sqrt{3}\rangle$ of $\mathbb{Z}[\sqrt{3}]$, respectively, cut into strips.

Let us start by considering Figures 4 and 5 again. Note that each of the graphs seem to contain points on the horizontal axis. If this were true for all such graphs, the location of these points could go a long way in helping us solidify our idea of density. To this end, consider that for any nonzero $a+b\sqrt{d}\in\mathbb{Z}[\sqrt{d}]$ we have

$$\begin{aligned}(a+b\sqrt{d})(a-b\sqrt{d}) &= a^2-db^2,\\ (a+b\sqrt{d})(-a+b\sqrt{d}) &= -(a^2-db^2).\end{aligned} \tag{1}$$

These are both integer elements of $\langle a+b\sqrt{d}\rangle$. Further, one must be positive and the other negative since d is square-free. Therefore, we can let n be the least positive integer such that $(n, 0)$ is in the graph associated with the ideal. Then it appears that the vertical strip of points between $(0, 0)$ and $(n, 0)$ in the graph repeats to the vertical strip between $(n, 0)$ and $(2n, 0)$, then again between $(2n, 0)$ and $(3n, 0)$, and so on throughout the entire plane. This generating strip in $\langle 3 + 1\sqrt{3}\rangle$ is thinner than the one in $\langle 3 + 1\sqrt{2}\rangle$. And while each strip contains a different number of points, copies of the strip in $\langle 3 + 1\sqrt{3}\rangle$ occur more often in the plane, yielding a more "dense" picture. So it seems our first step in determining density is finding the point $(n, 0)$. To this end, let us define dist(I) to be the least positive integer in $I = \langle a + b\sqrt{d}\rangle$.

Theorem 1. *Let $a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ be such that $a + b\sqrt{d} \neq 0$, and let $I = \langle a + b\sqrt{d}\rangle$. Then dist$(I)$ is given by*

1. $|db|$ *if* $a = 0$ *and* $b \neq 0$.
2. $|a|$ *if* $b = 0$ *and* $a \neq 0$.
3. $\frac{N(a+b\sqrt{d})}{\gcd(a,b)}$ *if* $a \neq 0$ *and* $b \neq 0$.

Proof. First recall that $\langle a + b\sqrt{d}\rangle$ contains at least one positive integer by (1).

Next, assume $a = 0$ and $b \neq 0$. Note that given any positive integer m such that $m \in \langle b\sqrt{d}\rangle$, there must exist an element $x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ such that

$$(b\sqrt{d})(x + y\sqrt{d}) = m.$$

This corresponds to having integer solutions to the system of equations

$$dby = m$$
$$bx = 0.$$

Note that the least possible positive value for m is obtained by setting $y = \pm 1$. Thus, $|db|$ is the smallest positive integer in $\langle a + b\sqrt{d}\rangle$. The case in which $a \neq 0$ and $b = 0$ is similarly proven.

Finally, assume $a \neq 0$ and $b \neq 0$. Then, given any positive integer $m \in \langle a + b\sqrt{d}\rangle$, there must exist an element $x + y\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ such that

$$(a + b\sqrt{d})(x + y\sqrt{d}) = m.$$

But this corresponds to having integer solutions to the system of equations

$$ax + dby = m \tag{2}$$
$$bx + ay = 0. \tag{3}$$

Let $g = \gcd(a, b)$, and let $a_1, b_1 \in \mathbb{Z}$ be such that $ga_1 = a$ and $gb_1 = b$. Substituting into (3) yields $gb_1x + ga_1y = 0$, or $a_1y = -b_1x$. Since a_1 and b_1 are relatively prime, there exists $p \in \mathbb{Z}$ such that $x = a_1p$ and $y = -b_1p$. Substituting into (2) yields

$$a(a_1p) + db(-b_1p) = m. \tag{4}$$

Substituting for a and b in (4) gives us

$$ga_1a_1p - dgb_1b_1p = m,$$

which can be written as

$$gp(a_1^2 - db_1^2) = m.$$

The smallest positive value for m is obtained when $p = \pm 1$. Thus,

$$g|a_1^2 - db_1^2| = \frac{g^2|a_1^2 - db_1^2|}{g} = \frac{|(ga_1)^2 - d(gb_1)^2|}{g} = \frac{|a^2 - db^2|}{g} = \frac{N(a + b\sqrt{d})}{\gcd(a, b)}$$

is the smallest positive integer in $\langle a + b\sqrt{d}\rangle$. ■

The generating square Setting $\operatorname{dist}(I) = n$, we notice that once the ordered pair $(n, 0)$ appears in an ideal, the pairs $(0, n)$ and (n, n) must also be present. Thus, the repeated pattern of "strips" that we saw in our earlier graphs can actually be described more precisely by squares. That is, given nonzero $a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$ and $I = \langle a + b\sqrt{d}\rangle$ with $\operatorname{dist}(I) = n$, we define the *generating square S of I* to be the square with corners

$$(0, 0), \quad (n - 1, 0), \quad (0, n - 1), \quad \text{and} \quad (n - 1, n - 1).$$

We can then rigorously define the *density*, $\rho(I)$, of an ideal I in $\mathbb{Z}[\sqrt{d}]$ with generating square S by the formula

$$\rho(I) = \frac{|I(S)|}{\operatorname{area}(S)}, \tag{5}$$

where $I(S)$ is the set of all elements of I that live within or on the boundary of S. See the highlighted portions of Figure 6 for reference.

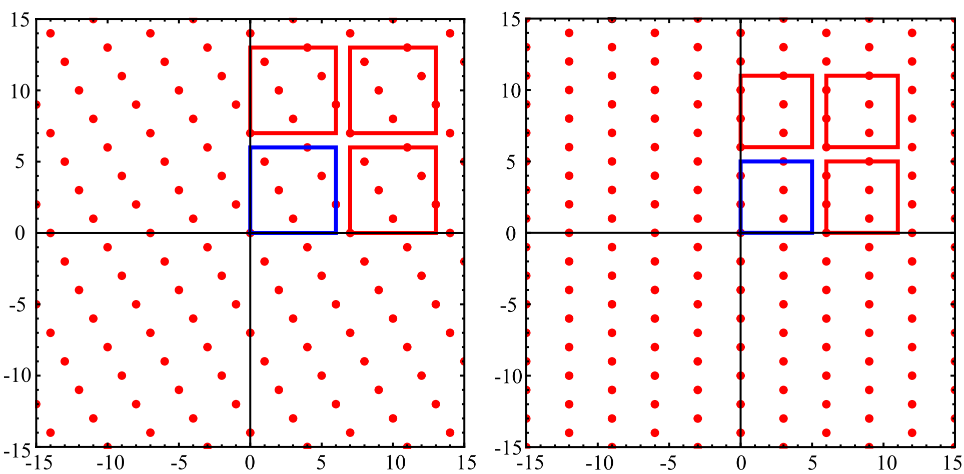

Figure 6 The generating square (and some of its copies) of $\langle 3 + 1\sqrt{2}\rangle$ and $\langle 3 + 1\sqrt{3}\rangle$, respectively.

Obviously, $\operatorname{area}(S) = n^2$ and $|I(S)| \leq n^2$. All that remains is to compute $|I(S)|$. Assume, for the moment, that we have two distinct ordered pairs in $I(S)$ with the same vertical coordinate. That is, assume that there exists $(u, w) \in I(S)$ and $(v, w) \in I(S)$ where $0 \leq u < v < n$ and $0 \leq w < n$. But then

$$(v, w) - (u, w) = (v - u, 0) \in I(S),$$

with $0 < v - u < n$, which contradicts that $\text{dist}(I) = n$. Thus, no two distinct elements of $I(S)$ have the same vertical coordinate. This means $|I(S)| \leq n$.

However, there may exist an integer $0 \leq w < n$ that is not the vertical coordinate of any element of $I(S)$. To find which integers $0 \leq w < n$ *are* vertical coordinates of points in $I(S)$, we recall that an ordered pair (v, w) is in $I(S)$ if and only if there exists $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$(a + b\sqrt{d})(x + y\sqrt{d}) = v + w\sqrt{d}.$$

This simplifies to finding integer solutions x and y to the equations

$$ax + dby = v \tag{6}$$

$$bx + ay = w. \tag{7}$$

Notice that (6) determines the horizontal value v and (7) determines the vertical value w, when thinking of (v, w) as a point in $I(S)$. When $a = 0$, (7) reduces to $bx = w$. So $I(S)$ must contain exactly one point with vertical coordinate w for each $0 \leq w < n$ such that $b|w$, and no other points. Since $n = |db|$ from Theorem 1, this gives a total of $\frac{n}{|b|} = |d|$ points in $I(S)$. A similar argument shows that when $b = 0$, there must be exactly $\frac{n}{|a|} = 1$ point in $I(S)$.

Finally, if $a \neq 0$ and $b \neq 0$, a well-known result in number theory states that (7) has integer solutions if and only if $\gcd(a, b)|w$. This fact, along with our earlier discussion, means that $I(S)$ contains exactly one point with vertical coordinate w for each $0 \leq w < n$ such that $\gcd(a, b)|w$, and no other points. Combining these results yields a total of

$$\frac{n}{\gcd(a, b)}, \qquad \text{or} \qquad \frac{N(a + b\sqrt{d})}{\gcd(a, b)^2}$$

from Theorem 1, points in $I(S)$.

Thus we have that the density of a nonzero ideal $I = \langle a + b\sqrt{d} \rangle$ in $\mathbb{Z}[\sqrt{d}]$ is given by

$$\rho(I) = \begin{cases} \dfrac{|d|}{n^2} = \dfrac{1}{N(b\sqrt{d})}, & \text{if } a = 0; \\[2ex] \dfrac{1}{n^2} = \dfrac{1}{N(a)}, & \text{if } b = 0; \\[2ex] \dfrac{\left(\frac{N(a+b\sqrt{d})}{\gcd(a,b)^2}\right)}{n^2} = \dfrac{1}{N(a + b\sqrt{d})}, & \text{if } a, b \neq 0. \end{cases}$$

Summarizing, we see that $\rho(I)$ always equals $\frac{1}{N(a+b\sqrt{d})}$. We could not have hoped for anything better! Not only have we discovered a wonderfully simple formula, but we have answered our question from earlier. Namely, it seems that norms (at least in our integral domains) are an inverse measure of the density of a principal ideal relative to the entire ring.

Theorem 2. *Let nonzero $a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$. Then the density of the principal ideal $\langle a + b\sqrt{d} \rangle$ in $\mathbb{Z}[\sqrt{d}]$ is given by $\frac{1}{N(a+b\sqrt{d})}$.*

Example. Remember how the ideal $\langle 3+1\sqrt{3}\rangle$ looked more "dense" than $\langle 3+1\sqrt{2}\rangle$ in Figure 5? According to Theorem 2 their densities are

$$\frac{1}{N(3+1\sqrt{3})}=\frac{1}{6} \qquad \text{and} \qquad \frac{1}{N(3+1\sqrt{2})}=\frac{1}{7},$$

respectively. So the ideal $\langle 3+1\sqrt{3}\rangle$ contains 1 out of every 6 elements of the ring $\mathbb{Z}[\sqrt{3}]$, while the ideal $\langle 3+1\sqrt{2}\rangle$ contains only 1 out of every 7 elements of the ring $\mathbb{Z}[\sqrt{2}]$. This confirms our initial observation.

Additional observations

By applying a visual approach to the study of integral domains of the form $\mathbb{Z}[\sqrt{d}]$, we discovered a connection between principal ideals and the norm of their generating element. This insight now allows us to state some very well-known properties of ring theory in a different, perhaps more descriptive, manner.

1. Since all units have a norm of 1, the density of the principal ideal they generate is $1/1$ and must be the entire ring.
2. Since the norm N of a nonzero element is always a positive integer, the cosets of a principal ideal must partition the ring into a set of N equivalence classes.
3. Similarly, there is no density ratio between $1/2$ and 1, hence the only principal ideal containing more than half the elements of a ring is the ring itself.

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Summary. In this paper we explore quadratic integer rings from a geometric point of view. We find that many important features of these structures have a useful representation in this alternate setting, giving further insight into their characteristics. An inverse relationship between the "density" of a principal ideal and the norm of its generator, in particular, is then shown to exist.

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Listing the Positive Rationals

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We start with a review of the usual method for listing the rationals. Consider an infinite array with the rows and columns indexed by the positive integers. The entry in the ith row and jth column is j/i. Each positive rational number occurs infinitely often in this array, but only once in lowest terms. The list of the rationals is then formed by moving along the diagonals from lower left to upper right including only the rational numbers that are in lowest terms. In Table 1, we show the upper corner of the array. Table 2 gives the first forty entries in this list.

	1	2	3	4	5	6	7	8	9	…
1	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{3}{1}$	$\frac{4}{1}$	$\frac{5}{1}$	$\frac{6}{1}$	$\frac{7}{1}$	$\frac{8}{1}$	$\frac{9}{1}$	…
2	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\frac{8}{2}$	$\frac{9}{2}$	…
3	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	$\frac{6}{3}$	$\frac{7}{3}$	$\frac{8}{3}$	$\frac{9}{3}$	…
4	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$	$\frac{5}{4}$	$\frac{6}{4}$	$\frac{7}{4}$	$\frac{8}{4}$	$\frac{9}{4}$	…
5	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{5}{5}$	$\frac{6}{5}$	$\frac{7}{5}$	$\frac{8}{5}$	$\frac{9}{5}$	…
6	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$	$\frac{7}{6}$	$\frac{8}{6}$	$\frac{9}{6}$	…
7	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{6}{7}$	$\frac{7}{7}$	$\frac{8}{7}$	$\frac{9}{7}$	…
8	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	$\frac{6}{8}$	$\frac{7}{8}$	$\frac{8}{8}$	$\frac{9}{8}$	…
9	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{6}{9}$	$\frac{7}{9}$	$\frac{8}{9}$	$\frac{9}{9}$	…
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

TABLE 1: The array of rational numbers

The 40th rational number in the list is $9/2$, and $2/9$ is 33rd in the list. To answer the question, "What rational number is 150th in the list?" we could simply construct the first 150 entries in the list. To answer the question, "Where is $21/13$ in the list?" we must construct the list until $21/13$ shows up. However, there is a short-cut to answering these questions based on the Euler phi function, $\phi(n)$, defined as the number of integers less than n that are relatively prime to n.

Lemma 1. *The positive integer r is relatively prime to the positive integer s if and only if r is relatively prime to $r + s$*

Proof. If $p > 1$ divides r and s, then it clearly divides $r + s$. If $p > 1$ divides r and $r + s$, then it clearly divides $(r + s) - r = s$. ■

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1	$\frac{1}{1}$	11	$\frac{5}{1}$	21	$\frac{7}{1}$	31	$\frac{9}{1}$
2	$\frac{1}{2}$	12	$\frac{1}{6}$	22	$\frac{1}{8}$	32	$\frac{1}{10}$
3	$\frac{2}{1}$	13	$\frac{2}{5}$	23	$\frac{2}{7}$	33	$\frac{2}{9}$
4	$\frac{1}{3}$	14	$\frac{3}{4}$	24	$\frac{4}{5}$	34	$\frac{3}{8}$
5	$\frac{3}{1}$	15	$\frac{4}{3}$	25	$\frac{5}{4}$	35	$\frac{4}{7}$
6	$\frac{1}{4}$	16	$\frac{5}{2}$	26	$\frac{7}{2}$	36	$\frac{5}{6}$
7	$\frac{2}{3}$	17	$\frac{6}{1}$	27	$\frac{8}{1}$	37	$\frac{6}{5}$
8	$\frac{3}{2}$	18	$\frac{1}{7}$	28	$\frac{1}{9}$	38	$\frac{7}{4}$
9	$\frac{4}{1}$	19	$\frac{3}{5}$	29	$\frac{3}{7}$	39	$\frac{8}{3}$
10	$\frac{1}{5}$	20	$\frac{5}{3}$	30	$\frac{7}{3}$	40	$\frac{9}{2}$

TABLE 2: The first forty reduced rationals

Theorem 1. *The number of reduced rational numbers in the ith diagonal is $\phi(i+1)$.*

Proof. The ith diagonal starts with $1/i$ and includes all fractions r/s where $r+s = i+1$. We note that r/s will be reduced if and only if r is relatively prime to s. But by Lemma 1, r is relatively prime to s if and only if r is relatively prime to $r+s$. Hence, $\phi(i+1)$ counts the number of reduced fraction in the ith diagonal. ■

The next useful fact is that $\phi(n)$ is given by a very nice formula:

$$\phi(n) = n \prod_{p \in \pi(n)} \left(1 - \frac{1}{p}\right),$$

where $\pi(n)$ denote the set of prime factors of n. The standard proof of this formula is based on the inclusion-exclusion formula and can be found in any elementary text on number theory or enumerative combinatorics. Putting this all together, we see that the number of rationals listed after working through the nth diagonal is the cumulative total of the values of $\phi(i+1)$ from 1 to n. These numbers, for i up to 18, are shown in Table 3.

The cumulative total of the values of $\phi(i+1)$ up to and including $i = 21$ is 149. Therefore, the 150th fraction in the list is $1/21$, the first reduced fraction in the 22nd diagonal. The fraction $21/13$ lies on the 33rd diagonal ($21 + 13 = 34$). We count the position of $21/13$ among the reduced fractions on that diagonal and add that number to 343, the cumulative total up to $n = 32$, giving us that $21/13$ is 353rd in this list.

The problem with this traditional approach is that there is no known, usable formula for the cumulative totals of the values of the Euler phi function. In fact, a formula for the inverse of this listing would have a formula for the cumulative totals of the values of the Euler phi function as a special case: the position of the integer $n-1$ on the list is the cumulative total of the Euler phi function up to $\phi(n)$. So, while not efficient, this traditional listing is still very interesting.

i	1	2	3	4	5	6	7	8	9	10
$\phi(i+1)$	1	2	2	4	2	6	4	6	4	10
Cum. tot.	1	3	5	9	11	17	21	27	31	41

i	11	12	13	14	15	16	17	18
$\phi(i+1)$	4	12	6	8	8	16	6	18
Cum. tot.	45	57	63	71	79	95	101	119

TABLE 3: Values of $\phi(i+1)$, and the cumulative total, for $1 \leq i \leq 18$

For simplicity and elegance, we recall Yoram Sagher's bijection between the positive integers and the positive rationals [4]. We can write the prime factorization of the positive integer n in the form $p_1^{e_1} \dots p_k^{e_k} q_1^{o_1} \dots q_h^{o_h}$ where the p_i represent the prime factors of n that appear to an even power and the q_j represent the prime factors of n that appear to an odd power. Define

$$f(n) = \frac{p_1^{\frac{e_1}{2}} \dots p_k^{\frac{e_k}{2}}}{q_1^{\frac{o_1+1}{2}} \dots q_h^{\frac{o_h+1}{2}}}.$$

To prove that this is a bijection, we simply give the formula for f^{-1}: if r and s are relatively prime, r has the prime factorization $p_1^{u_1} \dots p_k^{u_k}$, and s has the prime factorization $q_1^{v_1} \dots q_h^{v_h}$, where the p_i and the q_j are disjoint sets of primes, then one easily checks that

$$f^{-1}\left(\frac{r}{s}\right) = p_1^{2u_1} \dots p_k^{2u_k} q_1^{2v_1-1} \dots q_h^{2v_h-1}.$$

The 150th fraction in this list is easily computable: $150 = 2 \cdot 3 \cdot 5^2$ and so $f(150) = 5/6$. We also have

$$f^{-1}(21/13) = f^{-1}((3 \cdot 7)/13) = 3^2 \cdot 7^2 \cdot 13 = 5733.$$

While both the traditional method and that of Sagher efficiently prove that the rationals are countable, computing the actual values of their bijections involves prime factorization, and for sufficiently large integers, no efficient factorization algorithm is known. The approach we consider next avoids factorization and leads to a bijection with a straightforward algorithm for computing its values. The rest of this paper is devoted to developing this bijection, which is based on a very surprising method for constructing a list of positive rationals: the Calkin-Wilf sequence.

The Calkin-Wilf sequence

The Calkin-Wilf sequence first appeared in 2000 [2]. We introduce it here in the form given by Bates, Bunder, and Tognetti [1]. It is also discussed in the solution to a problem presented in the *American Mathematical Monthly* [3].

Definition 1 (Calkin-Wilf sequence). *Let $\ell(n)$ denote the nth term in the Calkin-Wilf sequence. Then*

$$\ell(n) = \begin{cases} \frac{1}{1} & \text{for } n = 1 \\ \frac{1}{\lfloor \ell(n-1) \rfloor + 1 - \{\ell(n-1)\}} & \text{for } n > 1 \end{cases}$$

where,for any rational number x, $\lfloor x \rfloor$ is the floor or integer part of x and $\{x\}$ is the fractional part of x.

The first few steps are:

$$\frac{1}{1} \to \frac{1}{1+1-0} = \frac{1}{2} \to \frac{1}{0+1-\frac{1}{2}} = \frac{2}{1} \to \frac{1}{2+1-0} =$$
$$\frac{1}{3} \to \frac{1}{0+1-\frac{1}{3}} = \frac{3}{2} \to \frac{1}{1+1-\frac{1}{2}} = \frac{2}{3} \to \dots$$

The key to understanding the Calkin-Wilf algorithm is the construction of the *Calkin-Wilf tree.*

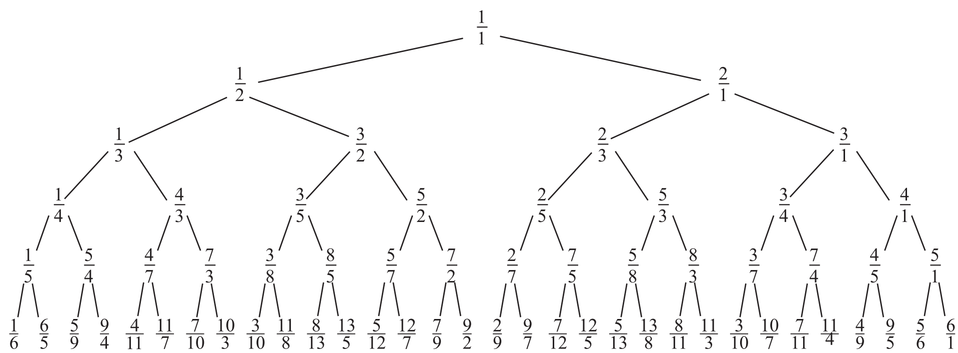

Figure 1 The first six levels of the Calkin–Wilf tree T_Q.

The Calkin–Wilf tree T_Q is an infinite, binary tree with vertices consisting of positive rational numbers represented as reduced fractions. It is defined inductively as follows: the root is $1/1$; the left child of p/q is $p/(p+q)$ and the right child of p/q is $(p+q)/q$. By Lemma 1, if p/q is reduced, both $p/(p+q)$ and $(p+q)/q$ are reduced. The first six generations are pictured above.

A similar tree first appeared in 1858 in a paper by M. A. Stern [5] and is often referred to as the Stern-Brocot Tree. The following two lemmas tell us first, that each positive rational number appears exactly once in the tree, and second, that the algorithm moves through this tree from left to right, one level at a time.

Lemma 2. *Each positive rational number appears exactly once in T_Q.*

Proof. We start by observing that 1/1 does appear in the tree as the root and, indeed, can only appear as the root. Our induction hypothesis is that every reduced fraction p/q with $p+q < n$ appears in the tree exactly once. Let r/s be any reduced fraction with $r+s = n$. If $r > s$ then $\frac{r}{s}$ can only be the right child of $(r-s)/s$. Since $(r-s)+s = r < n$, $(r-s)/s$ appears in the tree exactly once, and therefore r/s appears in the tree exactly once. Similarly, If $r < s$, then r/s can only be the left child of $r/(s-r)$. Since $r+(s-r) = s < n$, $r/(s-r)$ appears in the tree exactly once, and therefore r/s appears in the tree exactly once. ■

Lemma 3. *The integer $\ell(n+1)$ appears to the immediate right of $\ell(n)$ in T_Q, unless $\ell(n)$ is the right-most entry in a level. In that case, $\ell(n+1)$ is the left-most entry in the next level.*

Proof. Note that the ith level starts with $1/i$ and ends with $i/1$. There are two cases to consider. First, assume that $\ell(n) = i/1$. By direct computation

$$\ell(n+1) = \frac{1}{i+1-0},$$

which is the first entry in the next level.

Now suppose that $\ell(n)$ is not the right-most entry in a level, and let x denote the entry to its immediate right in the tree. Let p/q be the lowest vertex in the tree that is a common ancestor of $\ell(n)$ and x, and assume that p/q is $m+1$ levels above $\ell(n)$ and x. To get to $\ell(n)$ from p/q, we take the left child of p/q, which is $p/(p+q)$, and then m successive right children. To get to x from p/q, we take the right child of p/q, which is $(p+q)/q$, and then m successive left children. We have

$$\ell(n) = \frac{p+m(p+q)}{p+q} \qquad \text{and} \qquad x = \frac{p+q}{q+m(p+q)}.$$

Hence $\lfloor \ell(n) \rfloor = m$, $\{\ell(n)\} = p/(p+q)$, and

$$\ell(n+1) = \frac{1}{m+1-\frac{p}{p+q}} = \frac{p+q}{(m+1)(p+q)-p}$$
$$= \frac{p+q}{q+m(p+q)} = x$$

■

Combining Lemmas 3 and 4, we have

Theorem 2. *The list given by $\ell(n)$, for $n = 1, 2, \ldots,$ is a complete listing of all positive rational numbers without repeats.*

Remark 1. *Taking the root to be the first level, the ith level has 2^{i-1} entries. By induction, the first i levels of the Calkin-Wilf tree contain $2^i - 1$ entries and so $\ell(2^i) = 1/(i+1)$.*

Comparing the Calkin-Wilf sequence to the traditional list, ℓ is much easier to compute inductively. However, computing $\ell(150)$ inductively still requires 150 iterations, and computing the position of 21/13, an unknown number of iterations. Can we find a short-cut? The answer is yes. To compute the 150th entry, we note that $2^7 < 150 < 2^8$. Hence we start at 1/8 (the 128th) and iterate 22 times to obtain the 150th entry which is 18/25. This could still take some time for large values of n. However, we can do even better. Consider the binary counting number tree T_N, in which the vertices are the positive integers in the position obtained by simply counting the vertices, one level at a time, from left to right (Figure 2). Then $\ell(n)$ is the rational number occupying the position in T_Q corresponding to the position of n in T_N.

Taking a closer look at the structure of T_N, we note that the left child of the vertex n is simply $2n$, while its right child is $2n+1$. Hence, the binary representation of the left child of the vertex n is obtained from the binary representation of n by adding a 0 at its right end, and the binary representation of the right child of the vertex n is obtained from the binary representation of n by adding a 1 at its right end. This becomes clear

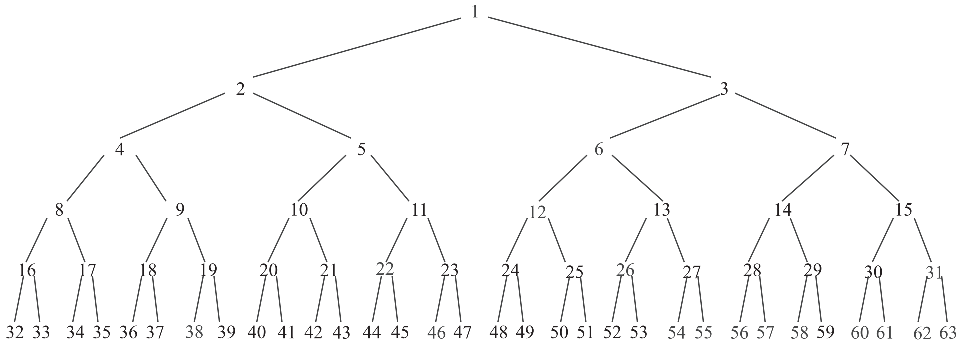

Figure 2 The first six levels of the binary counting tree T_N.

in the tree T_B, which is simply T_N with its vertices written in the binary number system (Figure 3).

One may think of the binary representation of n as the directions to its position. For example, 150 is 10010110 when written in binary. Starting at 1, the 0s in the second and third positions tell us to double 1, and to double it again, yielding 4. The next 1 tells us to double 4 and add 1 to get 9, and the following 0 doubles 9 to get 18.

Now doubling and adding 1 twice gives 37 and then 75. The last 0 instructs us to double 75 to get 150. Applying these same directions to T_Q, we can compute $\ell(150)$: starting at $1/1$ in the tree and the initial 1 in the binary expansion of 150, if a 0 follows move to the left child, if a 1 follows move to the right child. Thus, the next 0 directs us to $1/2$, the left child of $1/1$; 0 again gives $1/3$. Then in turn: 1 yields $4/3$, 0 gives $4/7$, 1 gives $11/7$, 1 gives $18/7$, and the final 0 gives $18/25$.

To answer the question, "Where does $21/13$ fit in the list?" we work backwards to construct the binary representation of its position number: $21/13$ is an improper fraction, so it is the right child of $8/13$ (that is, we have found a/b where $(a+b)/b = 21/13$, giving $a = 8$ and $b = 13$), so the last binary digit of its position is 1; $8/13$ is proper, so it is the left child of $8/5$, and therefore, the second last binary digit for $\frac{21}{13}$ is 0. This procedure is summarized in Table 4 (moving from right to left). Since 1010101 is the binary representation of 85, $\ell^{-1}(\frac{21}{13}) = 85$. Hence, we have a simple algorithm for computing the values of ℓ^{-1}. The function ℓ has several fascinating properties that the interested reader may wish to investigate. For example, note that

$$\ell\left(\left\lfloor \frac{2^8}{3} \right\rfloor\right) = \ell(85) = \frac{21}{13} = \frac{f_8}{f_7},$$

where f_i denotes the ith Fibonacci number.

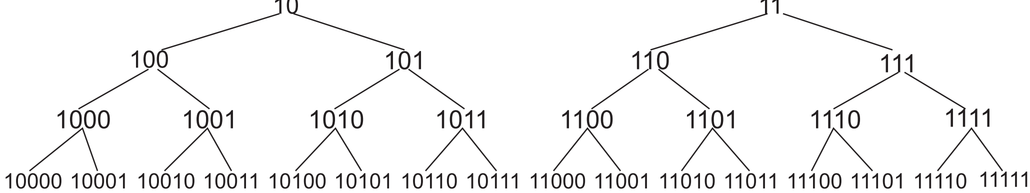

Figure 3 The first five levels of the binary counting tree T_B.

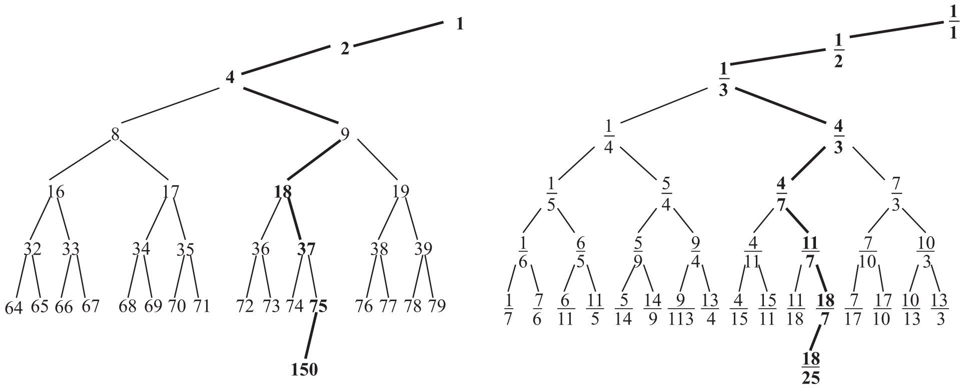

Figure 4 Computing $\ell(150)$.

1	0	1	0	1	0	1
$\frac{1}{1}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{8}{5}$	$\frac{8}{13}$	$\frac{21}{13}$

TABLE 4: The sequence of steps for finding where 21/13 appears in our list of rationals

Closed-form equation for $\ell(n)$

We conclude by giving a closed-form equation for $\ell(n)$. To do this we need another definition.

Definition 2 (Binary cipher)**.** *Let the binary representation of a positive integer n consist of a leading 1 followed by $a_{n,0}$ 0s, then $a_{n,1}$ 1s, then $a_{n,2}$ 0s, and so on up to $a_{n,k}$ 1s (for k odd) or $a_{n,k}$ 0s (for k even). We call*

$$\beta(n) = (a_{n,0}, a_{n,1}, \ldots, a_{n,k})$$

the binary cipher of n.

In the binary cipher of n, $a_{n,0}$ is nonnegative while the remaining entries are all positive. For example,

$$\beta(150) = (2, 1, 1, 2, 1) \qquad \text{and} \qquad \beta(12) = (0, 1, 2).$$

We may think of the binary cipher as directions to n in T_N. For example, starting at the root, we follow two edges to the left, one to the right, one to the left, two to the right and one to the left to arrive at 150. Using these same directions in the Calkin–Wilf tree T_Q, we get to $\ell(150)$. But to get there we need still more definitions.

Definition 3 (Odd binary product function)**.** *Let n be given with*

$$\beta(n) = (a_{n,0}, a_{n,1}, \ldots, a_{n,k}).$$

For all $i \leq k$, define $N_{n,i}$ to be the sum of 1 and all products of integers from $\{a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,i}\}$ such that the subscripts of components of each product are increasing, alternating in parity and ending with an odd subscript.

Definition 4 (Even binary product function)**.** *Let n be given with*

$$\beta(n) = (a_{n,0}, a_{n,1}, \dots, a_{n,k}).$$

For all $i \le k$, define $D_{n,i}$ to be the sum of 1 and all products of integers from $\{a_{n,0}, a_{n,1}, a_{n,2}, \dots, a_{n,i}\}$ such that the subscripts of components of each product are increasing, alternating in parity and ending with an even subscript.

We will soon prove that if

$$\beta(n) = (a_{n,0}, a_{n,1}, \dots, a_{n,k}),$$

then $N_{n,k}$ will be the numerator of $\ell(n)$ and $D_{n,k}$ its denominator. For example, if

$$\beta(n) = (a_{n,0}, a_{n,1}, a_{n,2}, a_{n,3}, a_{n,4}),$$

then

$$\begin{aligned}
N_{n,1} &= N_{n,2} = 1 + a_{n,1} + a_{n,0}a_{n,1},\\
N_{n,3} &= N_{n,4} = 1 + a_{n,1} + a_{n,0}a_{n,1} + a_{n,3} + a_{n,0}a_{n,3}\\
&\qquad + a_{n,2}a_{n,3} + a_{n,1}a_{n,2}a_{n,3} + a_{n,0}a_{n,1}a_{n,2}a_{n,3},\\
D_{n,1} &= 1 + a_{n,0},\\
D_{n,2} &= D_{n,3} = 1 + a_{n,0} + a_{n,2} + a_{n,1}a_{n,2} + a_{n,0}a_{n,1}a_{n,2},\\
D_{n,4} &= 1 + a_{n,0} + a_{n,2} + a_{n,4} + a_{n,1}a_{n,2}\\
&\qquad + a_{n,1}a_{n,4} + a_{n,3}a_{n,4} + a_{n,0}a_{n,1}a_{n,2} + a_{n,0}a_{n,1}a_{n,4}\\
&\qquad + a_{n,0}a_{n,3}a_{n,4} + a_{n,2}a_{n,3}a_{n,4} + a_{n,1}a_{n,2}a_{n,3}a_{n,4}\\
&\qquad + a_{n,0}a_{n,1}a_{n,2}a_{n,3}a_{n,4}.
\end{aligned}$$

So in the case of

$$\beta(150) = (2, 1, 1, 2, 1),$$

we have

$$\begin{aligned}
N_{150,4} &= 1 + 1 + 2 + 2 + 4 + 2 + 2 + 4 = 18,\\
D_{150,4} &= 1 + 2 + 1 + 1 + 1 + 1 + 2 + 2 + 2 + 4 + 2 + 2 + 4 = 25,
\end{aligned}$$

and we have already computed $\ell(150)$ to be $18/25$.

Lemma 4. *Let $\beta(n) = (a_{n,0}, a_{n,1}, a_{n,2}, \dots, a_{n,k})$ for the positive integer n, then, for $i \le k$:*

$$N_{n,i} = \begin{cases} N_{n,(i-1)}, & \text{if } i \text{ is even}\\ N_{n,(i-1)} + D_{n,(i-1)}a_{n,i}, & \text{if } i \text{ is odd}\end{cases}$$

$$D_{n,i} = \begin{cases} D_{n,(i-1)} + N_{n,(i-1)}a_{n,i}, & \text{if } i \text{ is even}\\ D_{n,(i-1)}, & \text{if } i \text{ is odd}\end{cases}$$

Proof. Consider $N_{n,i}$, and suppose that i is even. Then all products of the a_j with increasing indices alternating in parity and ending with an odd subscript less than or equal to i are included in $N_{n,(i-1)}$. and so

$$N_{n,i} = N_{n,(i-1)}.$$

Now assume that i is odd and and partition the terms of $N_{n(i)}$ in to those that end in a_i and those that do not. The sum of those that do not end in a_i is simply $N_{n,(i-1)}$ while the sum of those terms that do end in a_i is $D_{n,(i-1)}a_i$. The proof of the second equation is similar. ■

Theorem 3. *Let*

$$\beta(n) = (a_{n,0}, a_1, a_{n,2}, \ldots, a_{n,k})$$

for the positive integer n, then

$$\ell(n) = \frac{N_{n,k}}{D_{n,k}}.$$

Proof. We proceed by induction on k. Consider the case $k = 0$. That is, suppose that $\beta(n) = (a_{n,0})$. Then the binary representation of n is 1 followed by $a_{n,0}$ 0s, and so

$$N_{n,0} = 1 \qquad \text{and} \qquad D_{n,0} = 1 + a_{n,0}.$$

But $n = 2^{a_{n,0}}$ and, by Remark 1,

$$\ell(2^{a_{n,0}}) = \frac{1}{1 + a_{n,0}},$$

which does equal $\frac{N_{n,0}}{D_{n,0}}$. Thus, we have established the formula for the base case $k = 0$. We now proceed by strong induction on the length of the binary cipher.

Now let

$$\beta(n) = (a_{n,0}, a_1, a_{n,2}, \ldots, a_k),$$

and assume the formula holds for all m with shorter binary ciphers. Specifically, consider m where

$$\beta(m) = (a_{n,0}, a_1, a_{n,2}, \ldots, a_{k-1}).$$

Assume that $\ell(m) = p/q$ and that k is even. Then to get to $\ell(n)$ from $\ell(m)$, we take the left child of the left child of $\cdots$ of the left child of p/q, iterating $a_{n,k}$ times, giving

$$\ell(n) = \frac{p}{q + a_{n,k}p}.$$

If k had been odd, we would have iterated taking the right child of p/q $a_{n,k}$ times to get

$$\ell(n) = \frac{p + a_{n,k}q}{q}.$$

By the induction hypothesis,

$$\ell(m) = \frac{N_{m,(k-1)}}{D_{m,(k-1)}}.$$

That is,

$$p = N_{m,(k-1)} = N_{n,(k-1)} \qquad \text{and} \qquad q = D_{m,(k-1)} = D_{n,(k-1)}.$$

Hence,

$$\ell(n) = \begin{cases} \dfrac{N_{n,(k-1)}}{D_{n,(k-1)} + N_{n,(k-1)}a_k}, & \text{when } k \text{ is even,} \\[2ex] \dfrac{N_{n,(k-1)} + D_{n,(k-1)}a_k}{D_{n,(k-1)}}, & \text{when } k \text{ is odd.} \end{cases}$$

Finally, by Lemma 4,

$$\ell(n) = \frac{N_{n,k}}{D_{n,k}}.$$

■

Since we have a closed-form equation for $\ell(n)$, it is natural to ask if there is there a formula for $\ell^{-1}(n)$. This is just one of many intriguing questions, results and problems related to this listing of the rationals. The interested reader might start with the any of the references below or by searching the web for "Calkin-Wilf sequence" and "Stern-Brocot Tree."

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Summary. The rational numbers are countable. The traditional proof demonstrates that there exists a one-to-one function from the natural numbers onto the positive rational numbers or simply that there exists a list of all positive rationals (without repeats). But the list is not explicitly given. That is, there is no reasonable way to say which rational number is 150th in the list or where $21/13$ appears in the list. There are several different listings of the rationals, some much easier to compute, some involving interesting mathematical constructions. We start this paper with a discussion of the traditional method, then introduce the bijection that is easiest to define and finally concentrate on the intriguing Calkin-Wilf algorithm.

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Pythagorean Paper Folding

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> Even fairly good students, when they have obtained the solution of the problem and written down neatly the argument, shut their books and look for something else. Doing so, they miss an important and instructive phase of the work. By looking back at the completed solution, by reconsidering and reexamining the result and the path that led to it, they could consolidate their knowledge and develop their ability to solve problems. A good teacher should understand and impress on his students the view that no problem whatever is completely exhausted. There remains always something to do; with sufficient study and penetration, we could improve any solution, and, in any case, we can always improve our understanding of the solution.
>
> – George Pólya, *How to Solve It*

For several years, I was part of the *Focus on Mathematics* project, which partnered mathematicians and mathematics educators with teachers and students in Boston-area schools. My favorite responsibility with *FoM* was the chance to facilitate after-school study groups with teachers (I personally worked with elementary teachers in Arlington and Lawrence, middle school teachers in Waltham, and high school teachers in Chelsea). While each of these study group sessions provided endless satisfaction and shareable stories, one particular session forms the genesis of this note.

After my final meeting with the Chelsea High School study group, most of the study group participants met up at a local watering hole. One of the teachers, Mike Grubin, made a surprising claim about folding a unit square. If you fold the square so that its top right corner is placed at the midpoint of its left edge, then the triangles formed are all similar to the iconic 3-4-5 Pythagorean triangle. I have subsequently learned that this result is a special case of *Haga's theorem* ([5]).

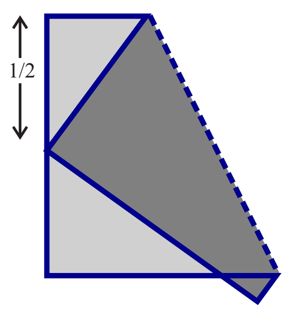

Figure 1 Folding the upper right corner to the left edge midpoint

Dubious as to whether the triangles would even have rational sides, I folded a square napkin as indicated and jotted down the dimensions of the three triangles that were formed (the two "obvious" ones are lightly shaded on the left side of Figure 1, while the third is located at the bottom right of the figure).

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As we shall see, showing that the three triangles are similar to the 3-4-5 triangle is fairly straightforward. After confirming the claim (see the next section for details), I wondered what would happen when a square's top right corner was folded to *another* rational point on its left edge. Would other Pythagorean triangles emerge? If so, which ones?

As I began to work on this, it occurred to me that my impulse to pursue these sort of "follow-up" questions is unfortunately extremely rare among the students I teach. Even the "best" tend to wait for *me* to give them a problem to work on rather than follow their own curiosity. The main thrust of this article is to pursue the generalized question and to ponder, in light of Pólya's quote, how we might encourage more students to perform similar "unassigned" investigations on their own.

I interpret the quote in two different ways. On the one hand, I see it as a call to ask the "next question(s)" that follow from the given one, but I also see it as a suggestion to look back at the work that's been done in order to learn more about the given context. To that end, I will attempt to trace the actual path I took in solving this problem (and the related consequences) rather than supplying the traditional "efficient" path usually described in a text or article (although I *will* leave some calculations to the reader). I will also attempt to make clear when "looking back" provided me with useful insights into the problems. It is my hope that this approach will tell a more authentic story of the problem-solving process which I feel is too often left hidden by more traditional expositions. As my friend and colleague, Paul Goldenberg says, "A good proof doesn't just confirm THAT something is true, it reveals WHY it's true." Answering "why" is my ultimate goal.

Confirming the claim. I'll proceed here much as I did originally, with the added benefit of better graphics. As described above, folding the square so that its top right corner is moved to the midpoint of its left edge, we create three triangles, labeled *ABC*, *DCE*, and *FGE*, as shown in Figure 2.

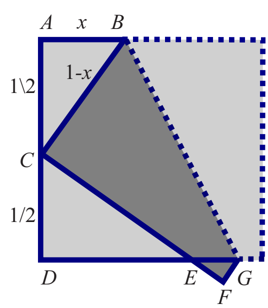

Figure 2 Fold to the midpoint (with labels)

We'll first show that $\triangle ABC$ is similar to the 3-4-5 right triangle. Setting $AB = x$, we have $BC = 1 - x$ (since $AB + BC = 1$, the length of each side of the original, unfolded square). Since $\angle BAC$ is a right angle (being a vertex angle of the original square), the Pythagorean theorem gives us

$$\left(\frac{1}{2}\right)^2 + x^2 = (1-x)^2.$$

Solving for x, we have $x = \frac{3}{8}$ and $1 - x = \frac{5}{8}$, and we see that $\triangle ABC$, having side lengths $\frac{3}{8}$, $\frac{1}{2}$, and $\frac{5}{8}$ is similar to the 3-4-5 right triangle.

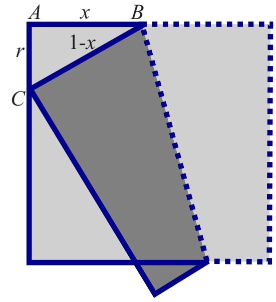

Figure 3 Fold to the point r from the top

Confirming that $\triangle DCE$ and $\triangle FGE$ are also similar to a $3-4-5$ triangle can be done in a number of ways. We could proceed analogously to the above argument and show that

$$CD = \frac{1}{2}, \quad DE = \frac{2}{3}, \quad CE = \frac{5}{6}, \quad FG = \frac{1}{8}, \quad EF = \frac{1}{6}, \quad \text{and} \quad EG = \frac{5}{24},$$

but a more efficient method is to observe that

$$\triangle ABC \sim \triangle DCE \sim \triangle GFE$$

by the angle-angle similarity theorem: $\angle CDE$ and $\angle FGE$ are congruent right angles since they are vertex angles of the square, $\angle CED$ and $\angle GEF$ are congruent vertical angles, while the pair $\angle ACB$ and $\angle DCE$ are complementary angles (as they, along with the right $\angle BCE$ form a "linear triple") confirming that $\angle BCA \cong \angle CED$.

Generalization from $\frac{1}{2}$ to any rational point. Fascinated by the fact (and surprised I had never run across it before), I wondered what would happen if the top right corner was folded to *another* rational point on the left side. Not surprisingly, the general case—what happens when the upper right corner of the square is folded to the point that is r below the top of the left edge?—can be approached similarly to the case when $r = \frac{1}{2}$.

First, note that the argument showing that the three right triangles are similar to one another was not dependent on the assumption $r = \frac{1}{2}$. Folding the upper right corner to the point on the left edge that is a distance of r below the top left corner gives rise to the triangle shown in the upper left of Figure 3. Working with $\triangle ABC$, as before, and letting $AB = x$ (and $BC = 1 - x$), we see that $r^2 + x^2 = (1-x)^2$, so $2x = 1 - r^2$. Therefore,

$$AB = \frac{1-r^2}{2}, \qquad AC = r, \qquad \text{and} \qquad BC = \frac{1+r^2}{2}.$$

We may show (using the same argument as in the case $r = \frac{1}{2}$) that the other two triangles are similar to $\triangle ABC$, but they are also rational. Leaving the computational details to the reader, the lower left triangle has dimensions

$$1 - r, \qquad \frac{2r}{1+r}, \qquad \text{and} \qquad \frac{1+r^2}{1+r},$$

while the lower right triangle has dimensions

$$\frac{(1-r)^2}{2}, \qquad r\left(\frac{1-r}{1+r}\right), \qquad \text{and} \qquad \frac{(1+r^2)(1-r)}{1+r}.$$

Summarizing what we've learned so far, if the upper right corner of the unit square is folded to a rational point along the left edge of the square, the resulting triangles are rational triangles, each of which is therefore similar to some Pythagorean triangle. While I found this to be a surprising fact, it led me to yet another question—does every Pythagorean triangle correspond to a triangle created by folding?

Which Pythagorean triangles can be generated by folding?

We are interested in rational values of r, the location of the square's top right corner relative to its top left corner after folding. The triangle labeled $\triangle ABC$ in Figure 3 has side lengths

$$AB = \frac{1-r^2}{2}, \qquad AC = r, \qquad \text{and} \qquad BC = \frac{1+r^2}{2}.$$

Substituting $r = \frac{a}{b}$ (where a and b are natural numbers), we have

$$AB = \frac{b^2-a^2}{2b^2}, \qquad AC = \frac{a}{b}, \qquad \text{and} \qquad BC = \frac{b^2+a^2}{2b^2}.$$

Multiplying each of these lengths by $2b^2$, we see that when the upper right corner of the unit square is folded to a point $\frac{a}{b}$ below the top of the left edge, the three triangles that are created are similar to the Pythagorean triangle with integral side lengths b^2-a^2, $2ab$, and b^2+a^2. It is important to note here that r must be strictly less than 1, since folding the top right corner of the unit square to the bottom left point of the square (1 unit below the top left corner) leads to a single triangle (not three triangles) with side lengths 1, 1, and $\sqrt{2}$, which is not similar to any Pythagorean triangle since $\sqrt{2}$ is irrational.

The integral lengths b^2-a^2, $2ab$, and b^2+a^2 will likely look familiar to readers who have studied Pythagorean triples in a number theory course or elsewhere (see, for example, Benson et al. [**1**, pp. 139–145] or Burton [**2**, pp. 247–249]). When I realized that folding the top right point to the point $\frac{a}{b}$ from the top of the left edge resulted in rational triangles similar to the Pythagorean triangle with side lengths b^2-a^2, $2ab$, and b^2+a^2, I wondered whether I had stumbled upon an alternative method of generating all Pythagorean triangles (up to similarity) and decided to proceed in that direction, pretending I didn't know about the Pythagorean triangle formula.

First, we need two definitions to help with the subsequent discussion.

Definition 1. *A Pythagorean triangle is* foldable *if it is similar to (the top left) triangle formed when the top right corner of the unit square is folded to the point $\frac{a}{b}$ from the top of the square's left edge.*

Definition 2. *A* foldable triple *is a triple representing the side lengths of $\triangle ABC$ when the top right corner of a unit square is folded to a rational distance below the top of the left edge of the square.*

The above discussion is summarized by the following Lemmas:

Lemma 1. *A Pythagorean triangle is foldable if it is similar to the triangle with side lengths b^2-a^2, $2ab$, and b^2+a^2 for some natural numbers a and b.*

Lemma 2. *Each foldable triple is of the form*

$$\left(\frac{b^2-a^2}{2b^2}, \quad \frac{a}{b}, \quad \frac{b^2+a^2}{2b^2}\right),$$

where a and b are positive integers with $a < b$. In particular, each triple of this form is similar to the Pythagorean triple $(b^2 - a^2, 2ab, b^2 + a^2)$ and a foldable triple can be expressed in the form (s, r, t), where s, r, and t are rational numbers between 0 and 1 satisfying $r^2 + s^2 = t^2$ and $r + t = 1$.

As demonstrated above, foldable triples are of the form

$$\left(\frac{b^2 - a^2}{2b^2}, \frac{a}{b}, \frac{b^2 + a^2}{2b^2}\right) = \left(\frac{b^2 - a^2}{2b^2}, \frac{2ab}{2b^2}, \frac{b^2 + a^2}{2b^2}\right),$$

where a and b are positive integers with $a < b$. This triple is similar to the Pythagorean triple $(b^2 - a^2, 2ab, b^2 + a^2)$, so every foldable triple is similar to a Pythagorean triple.

The above discussion really just provides another way to view what we already knew—that every rational fold leads to a triangle that is similar to a Pythagorean triangle. Does it help us answer the previously posed follow-up question? Using our new vocabulary, is every Pythagorean triple foldable? If (x, y, z) is a Pythagorean triple, to which foldable triple is (x, y, z) similar?

Basically, we need to find natural numbers a and b so that (x, y, z) is a rational multiple of either $(b^2 - a^2, 2ab, b^2 + a^2)$ or $(2ab, b^2 - a^2, b^2 + a^2)$. The reader might surmise that it would suffice to find a and b so that (x, y, z) *equals* either $(b^2 - a^2, 2ab, b^2 + a^2)$ or $(2ab, b^2 - a^2, b^2 + a^2)$. However, the Pythagorean triple (9, 12, 15) illustrates the problem with this way of thinking, since $b^2 - a^2 = 9$ for non-negative integers only when $b = 5$ and $a = 4$ (but then $2ab = 40$) or $b = 3$ and $a = 0$ (in which case $2ab = 0$). But (9, 12, 15) is certainly a foldable triple, since it is similar to (3,4,5), which is similar to $(\frac{3}{8}, \frac{1}{2}, \frac{5}{8})$, the very triple that motivated this investigation!

Back to determining whether each Pythagorean triple is similar to a foldable triple, how could I show that a given Pythagorean triple was a rational multiple of either $(b^2 - a^2, 2ab, b^2 + a^2)$ or $(2ab, b^2 - a^2, b^2 + a^2)$? Should I introduce a rational multiple into the triple and attempt to solve three equations for these three unknowns? In an attempt to find a method that avoided solving such a system, I decided to investigate the nature of foldable triples more closely.

Each foldable triple looks like

$$\left(\frac{b^2 - a^2}{2b^2}, \quad \frac{a}{b}, \quad \frac{b^2 + a^2}{2b^2}\right),$$

each of the terms is less than 1 and the first and third terms in the triple sum to 1. Of course, the final observation is not surprising, since the first and third terms represent AB and BC in the original folded triangle $\triangle ABC$, where $\overline{AB}$ and $\overline{BC}$ together form the top of the original (unfolded) unit square. Focusing on the first and third terms adding to 1, I decided to look at the rational triple

$$\left(\frac{x}{x+z}, \quad \frac{y}{x+z}, \quad \frac{z}{x+z}\right),$$

which is similar to our Pythagorean triple (x, y, z). Notice that setting $a = y$ and $b = x + z$ gives us $\frac{a}{b} = \frac{y}{x+z}$,

$$\begin{aligned}\frac{b^2 - a^2}{2b^2} &= \frac{(x+z)^2 - y^2}{2(x+z)^2} = \frac{x^2 + 2xz + z^2 - (z^2 - x^2)}{2(x+z)^2} \\ &= \frac{2x^2 + 2xz}{2(x+z)^2} = \frac{2x(x+z)}{2(x+z)^2} = \frac{x}{x+z}\end{aligned}$$

and

$$\frac{b^2+a^2}{2b^2} = \frac{(x+z)^2+y^2}{2(x+z)^2} = \frac{x^2+2xz+z^2+z^2-x^2}{2(x+z)^2}$$
$$= \frac{2z^2+2xz}{2(x+z)^2} = \frac{2z(x+z)}{2(x+z)^2} = \frac{z}{x+z}.$$

Note that the triangle inequality guarantees that $x+z>y$, so $\frac{y}{x+z}<1$, and therefore

$$\left(\frac{x}{x+z}, \frac{y}{x+z}, \frac{z}{x+z}\right) = \left(\frac{(x+z)^2-y^2}{2(x+z)^2}, \frac{y}{x+z}, \frac{(x+z)^2+y^2}{2(x+z)^2}\right)$$

is a foldable triple that is similar to (x, y, z), confirming the following theorem:

Theorem 1. *Every Pythagorean triple is similar to a foldable triple.*

In fact, we have provided an alternative (paper-folding) proof of the theorem stating that every Pythagorean triple is similar to a triangle of the form $(b^2-a^2, 2ab, a^2+b^2)$.

Again, in the spirit of Pólya, I decided to continue asking questions, even though, I had solved my original problem. I wondered how many different (reduced) fractions generated a given Pythagorean triangle (up to similarity).

Can two different fractions generate similar triangles?

My ability to determine which fractions generated similar triangles was due to a combination of perseverance and the good luck of noticing the right pattern. Using the notation $\frac{a}{b} \sim \frac{c}{d}$ to denote that the foldable triples generated by these fractions are similar, I found that $\frac{1}{2} \sim \frac{1}{3}$ by starting with the already established fact that $\frac{1}{2}$ generated the Pythagorean triple (3, 4, 5). Recalling that this came from the foldable triple

$$\left(\frac{3}{3+5}, \frac{4}{3+5}, \frac{5}{3+5}\right) = \left(\frac{3}{8}, \frac{1}{2}, \frac{5}{8}\right),$$

I considered the triple (4, 3, 5), leading me to the foldable triple

$$\left(\frac{4}{4+5}, \frac{3}{4+5}, \frac{5}{4+5}\right) = \left(\frac{4}{9}, \frac{1}{3}, \frac{5}{9}\right),$$

which is created when you fold the top right corner to the $\frac{1}{3}$ point. I performed a similar transformation with $\frac{1}{4}$ (corresponding to the foldable triple $\left(\frac{3}{5}, \frac{8}{25}, \frac{17}{25}\right)$, generating the triple (15, 8, 17), which is similar to the triple (8, 15, 17) and which corresponds to the foldable triple $\left(\frac{8}{25}, \frac{3}{5}, \frac{17}{25}\right)$, showing that $\frac{1}{4} \sim \frac{3}{5}$. Wondering if there was some sort of numerical connection between these similar rational numbers, I made two observations about each similar pair of reduced fractions (acknowledging that I was working with limited data, so would likely need to investigate further):

- One fraction had an odd numerator and denominator, while the other had numerator and denominator of opposite parity (in fact, in the examples so far, the numerator was odd and the denominator was even).
- If one of the fractions was $\frac{a}{b}$, the other was equivalent to $\frac{b-a}{b+a}$.

I then checked that $\frac{1}{6} \sim \frac{5}{7}$ (since these two fractions generated the triples (35, 12, 37) and (12, 35, 37), respectively). Doubting that in the case where the numerator and denominator had different parity the numerator would always be odd, I decided to check whether $\frac{2}{3}$ was similar to $\frac{3-2}{3+2} = \frac{1}{5}$ and found that $\frac{2}{3} \sim \frac{1}{5}$, since these fractions generated the triples (5, 12, 13) and (12, 5, 13), respectively. Believing I was on to something, I confirmed that $\frac{2}{5} \sim \frac{3}{7}$, $\frac{3}{8} \sim \frac{5}{11}$, and $\frac{7}{16} \sim \frac{9}{23}$, then I worked to prove the following theorem:

Theorem 2. *Two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, generate similar foldable triangles if and only if $\frac{a}{b} = \frac{c}{d}$ or $\frac{a}{b} = \frac{d-c}{d+c}$.*

Proof. It's *very* useful to first notice that

$$\frac{2ab}{(b^2-a^2)+(b^2+a^2)} = \frac{2ab}{2b^2} = \frac{a}{b},$$

as long as $b \neq 0$. If $\frac{a}{b}$ and $\frac{c}{d}$ generate similar foldable triangles, then the triangle having sides of length $b^2 - a^2$, $2ab$, and $b^2 + a^2$ is similar to the triangle with side lengths $d^2 - c^2$, $2cd$, and $d^2 + c^2$, and therefore either

$$(b^2-a^2, 2ab, b^2+a^2) = k(d^2-c^2, 2cd, d^2+c^2)$$

or

$$(b^2-a^2, 2ab, b^2+a^2) = k(2cd, d^2-c^2, d^2+c^2)$$

for some nonzero rational number k. In the first case,

$$\begin{aligned}\frac{a}{b} = \frac{2ab}{(b^2-a^2)+(b^2+a^2)} &= \frac{2cdk}{k(d^2-c^2)+k(d^2+c^2)} \\ &= \frac{2cd}{(d^2-c^2)+(d^2+c^2)} = \frac{c}{d}\end{aligned}$$

and in the second case,

$$\begin{aligned}\frac{a}{b} = \frac{2ab}{(b^2-a^2)+(b^2+a^2)} &= \frac{k(d^2-c^2)}{2cdk+k(d^2+c^2)} \\ &= \frac{k(d^2-c^2)}{k(d+c)^2} = \frac{d^2-c^2}{(d+c)^2} = \frac{d-c}{d+c},\end{aligned}$$

and the theorem is proved. ■

Since

$$\frac{(b+a)-(b-a)}{(b+a)+(b-a)} = \frac{a}{b},$$

a consequence of our theorem is that $\frac{a}{b}$ and $\frac{b-a}{b+a}$ are the only two unequal rational numbers that will generate a triple similar to $(b^2-a^2, 2ab, b^2+a^2)$ or $(2ab, b^2-a^2, b^2+a^2)$ (this is because $\frac{a}{b} = \frac{b-a}{b+a} \leftrightarrow a^2 + 2ab - b^2 = 0 \leftrightarrow a = b(\sqrt{2}-1)$, implying $\frac{a}{b} = \sqrt{2} - 1$ is irrational).

It is also interesting to note that the rational numbers $\frac{a}{b}$ and $1 - \frac{a}{b} = \frac{b-a}{b}$ do *not* generate similar triangles (unless, of course, $\frac{a}{b} = \frac{1}{2}$), even though one of the triangles created by folding the top right corner of the unit square to the point $\frac{a}{b}$ below the top left corner of the square has a length of $\frac{b-a}{b}$. In fact, if the top right corner of the square is folded to the point $\frac{b-a}{b}$ below the square's top left corner, then the resulting triangles will not be similar to the triangles formed when the the corner is folded to the point $\frac{a}{b}$ below the top left corner of the square. We leave the proof of this fact to the reader.

Of course, if c and d are both odd, then $d-c$ and $d+c$ will both be even, and therefore $\frac{1}{2}(d+c)$ and $\frac{1}{2}(d-c)$ are integers. The fact that c and d are odd guarantees each is either 1 more or 1 less than a multiple of 4. If c and d are congruent $\pmod 4$, then $\frac{1}{2}(d+c)$ is odd and $\frac{1}{2}(d-c)$ is even. Otherwise, $\frac{1}{2}(d+c)$ is even and $\frac{1}{2}(d-c)$ is odd. Furthermore, any common integer factor of $\frac{1}{2}(d+c)$ and $\frac{1}{2}(d-c)$ will also be a common divisor of $d=\frac{1}{2}(d+c)+\frac{1}{2}(d-c)$ and $c=\frac{1}{2}(d+c)-\frac{1}{2}(d-c)$. So, if c and d are relatively prime, then so are $\frac{1}{2}(d+c)$ and $\frac{1}{2}(d-c)$, and we have the following result:

Lemma 3. *Let c and d be relatively prime integers*

i. If c and d are both odd, then $\frac{1}{2}(d+c)$ and $\frac{1}{2}(d-c)$ will be relatively prime and *exactly one of them will be odd and the other will be even.*

ii. If exactly *one of c and d are odd, then $d+c$ and $d-c$ are relatively prime and both are odd.*

Our penultimate theorem is well known to students of number theory, but as far as I can tell, this "proof by folding" is unique.

Theorem 3. *A triple of positive integers is a primitive Pythagorean triple if and only if it is of the form $(b^2-a^2, 2ab, b^2+a^2)$ where a and b are natural numbers such that $a<b$, $gcd(a,b)=1$, and exactly one of a or b is odd.*

Proof. Recall that a primitive Pythagorean triple is a triple (x, y, z) of positive integers satisfying $x^2+y^2=z^2$, withd x, y, and z pairwise relatively prime. It is straightforward to show that every Pythagorean triple is a positive integer multiple of a *primitive* Pythagorean triple. Given the primitive triple (x, y, z), we know that x and y must be relatively prime. The proof of Theorem 1, along with Theorem 2, shows that the triangle with sides (x, y, z) is foldable, with fold point either $\frac{x}{y}$ or $\frac{y-x}{y+x}$ (with relatively prime numerator and denominator). Lemma 3 guarantees that exactly one of these two fold points has a numerator and denominator with opposite parity. Let $a=x$ and $y=b$ if x and y are both odd (otherwise, let $a=y-x$ and $b=y+x$). We know that our original primitive triple (x, y, z) is similar to $(\frac{b^2-a^2}{2b^2}, \frac{a}{b}, \frac{b^2+a^2}{2b^2})$, which is similar to the primitive triple $(b^2-a^2, 2ab, b^2+a^2)$, where a and b are relatively prime, $a<b$, and exactly one of a or b is even, completing the proof of Theorem 3. ■

The proof of Theorem 3 is a consequence of choosing fold points for which the numerator and denominator have opposite parity. That choice was not arbitrary, since I knew that choice would lead to the familiar formula, which was my initial goal. However, I could have just as easily chosen the rational numbers for which both numerator and denominator are odd.

So, what happens if we generate primitive Pythagorean triples with fold points having odd numerator and denominator (instead of those with different parity)? In this case, our reasoning continues exactly as in Theorem 3, with the alternative choice for a and b. This gives rise to the following theorem that provides an alternative formula for primitive Pythagorean triples with different restrictions.

Theorem 4. *A positive integer triple is a primitive Pythagorean triple if and only if it is of the form* $(\frac{b^2-a^2}{2}, ab, \frac{b^2+a^2}{2})$, *where a and b are relatively prime odd natural numbers with* $a < b$.

Note that the triples in Theorem 4 are not merely half of the triples in Theorem 3, since in Theorem 3, one of a and b is odd and the other is even, which is not the case in Theorem 4. Of course this is true since if you halve the triples from Theorem 3, only one of the resulting entries (namely ab) will be an integer, while if a and b are both odd, all three of $\frac{b^2-a^2}{2}$, ab, and $\frac{b^2+a^2}{2}$ will be integers. That every triple of the stated form is a Pythagorean triple is easily checked, while the fact that these are primitive triples is due to Lemma 3.

Concluding thoughts

I wanted to share these results (along with the process used) in the hopes that it might encourage others, especially students, to engage in "asking the *next* question". You never know where it might lead, which is one of the things I love about mathematics and problem solving. I pursue these types of questions because I'm compelled to do so. I *have* to know *why* a given statement is true. Trying to answer "why" is something that is part of me—I need to make sense of what's going on.

As all teachers know, this compulsion to ask and solve unassigned questions is not automatic with all of our students. It is the rare student that follows Pólya's advice without being assigned to do so, but this might be due to a lack of experience with and opportunities to pursue open-ended questions. Like-minded colleagues in the Inquiry Based Learning community have created materials and organized workshops and conferences to assist faculty in introducing this type of inquiry into mathematics courses at all levels (see Ernst, Hodge, and Yoshinobu [**3**] for more information and resources). It is my hope that this activity, among others, might spur interested students to make their own discoveries and experience the joy of doing mathematics independent from a class assignment.

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Summary. We start with a claim involving the triangles formed when the top right corner of a square is folded to the midpoint of the square's left side. Motivated by a quote from George Polya, the author "follows his nose" to make some interesting follow-up discoveries, including a new way to generate all primitive Pythagorean triples.

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Sums of Powers Via Matrices

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Power sums of integers have always been fascinating for professional mathematicians and amateurs alike. There is something intriguing about these sums, a mixture of complexity and simplicity that helps keep the subject alive. Over the years, a vast amount of research has been accumulated [1–6, 8–12]. In this article, we use matrices to exploit difference sequences and their corresponding difference tables. We prove a known formula for the sums of powers of consecutive natural numbers involving the Stirling numbers of the second kind, as well as additional results about difference sequences.

Formulas (1) and (2) are encountered in many introductory mathematics courses

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \tag{1}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \tag{2}$$

and are relatively simple to derive. However, the generalization to higher powers given by (3) and (4),

$$\sum_{k=1}^{n} k^d = \sum_{k=1}^{d} \binom{n+1}{k+1} k! \left\{ {d \atop k} \right\} \tag{3}$$

$$\sum_{k=1}^{n} k^d = \frac{1}{d+1} \sum_{k=0}^{d} \binom{d+1}{k} B_k n^{d+1-k} \tag{4}$$

require the understanding of Stirling numbers of the second kind and Bernoulli numbers, respectively. A quick glance at (4) shows that the formula for the sum of d powers of the first n natural numbers is a polynomial of degree $d+1$ in n. Note that $B_0 = 1$ is the first Bernoulli number (see Beardon [1], and Graham, Knuth and Patashnik [7]). For a similar formula involving Eulerian numbers see Krishnapniyan [10].

In this article, we are going to derive a variation of formula (3) via difference tables and matrices.

Preliminary results

Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. We define its first order difference sequence, $(\Delta a_n)_{n\geq 0}$, by $\Delta a_n = a_{n+1} - a_n$. For $k \geq 1$ and $n \geq 0$, we define the kth order difference $\Delta^k a_n = \Delta\left(\Delta^{k-1} a_n\right)$ and for $k = 0$ we define $\Delta^0 a_n = a_n$.

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For example, for $k = 2$ we have

$$\Delta^2 a_n = \Delta(a_{n+1} - a_n) = a_{n+2} - a_{n+1} - a_{n+1} + a_n = a_{n+2} - 2a_{n+1} + a_n,$$

and for $k = 3$ we have

$$\begin{aligned}\Delta^3 a_n = \Delta\left(\Delta^2 a_n\right) &= \Delta(a_{n+2} - 2a_{n+1} + a_n) \\ &= a_{n+3} - 2a_{n+2} + a_{n+1} - a_{n+2} + 2a_{n+1} - a_n \\ &= a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n\end{aligned}$$

The sequence $(a_n)_{n\geq 0}$, along with its difference sequences, are listed in the difference table below.

Difference Table 1

$$\begin{array}{cccccccccccccc}
a_0, & & a_1, & & a_2, & & a_3, & & a_4, & & a_5, & & \dots \\
& \Delta a_0, & & \Delta a_1, & & \Delta a_2, & & \Delta a_3, & & \Delta a_4, & & & \dots \\
& & \Delta^2 a_0, & & \Delta^2 a_1, & & \Delta^2 a_2, & & \Delta^2 a_3, & & & & \dots \\
& & & \Delta^3 a_0, & & \Delta^3 a_1, & & \Delta^3 a_2, & & & & & \dots \\
& & & & \ddots & & \ddots & & \ddots & & & & \dots
\end{array}$$

We now consider the sequence $0, 1, 3, 6, 10, 15, \dots$, which is a sequence starting with 0 followed by the partial sums of the natural numbers, along with its difference table.

Difference Table 2

$$\begin{array}{cccccccccccccccc}
0, & & 1, & & 3, & & 6, & & 10, & & 15, & & 21, & & \dots \\
& 1, & & 2, & & 3, & & 4, & & 5, & & 6, & & \dots \\
& & 1, & & 1, & & 1, & & 1, & & 1, & & \dots \\
& & & 0, & & 0, & & 0, & & 0, & & \dots
\end{array}$$

Difference table 2 was constructed starting with the top row. Clearly, the difference table can also be constructed using the entries of the 0th diagonal. The 0th diagonal consists of the entries $\Delta^k a_0$, $k = 0, 1, 2, 3$ (indexing starts with zero for both rows and diagonals). More precisely, and ignoring the last row of zeros, the entire difference table can be generated, one diagonal at a time, by multiplying the 0th diagonal vector $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^t$ by powers of the matrix

$$T_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

More precisely,

$$\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix},\quad \begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}^2\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}3\\3\\1\end{bmatrix}$$

$$\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}^3\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}3\\3\\1\end{bmatrix}=\begin{bmatrix}6\\4\\1\end{bmatrix}\dots$$

$$\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}^n\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}\sum_{k=1}^n k\\1+n\\1\end{bmatrix} \tag{5}$$

As long as the difference table of a sequence contains only zeros in its $(d+2)$th row, (i.e. $\Delta^{d+2}a_n = 0$ for all n), we can use the $(d+2)\times(d+2)$ matrix

$$T_d = \begin{bmatrix}1&1&0&0&\dots&0\\0&1&1&0&\dots&0\\0&0&1&1&\ddots&\vdots\\0&0&0&1&\ddots&0\\\vdots&\vdots&\vdots&\ddots&\ddots&1\\0&0&0&\dots&0&1\end{bmatrix}$$

to reconstruct it, assuming we know its 0th diagonal.

Proposition 1. *Let n and d be nonnegative integers. Then*

$$T_d^n = \begin{bmatrix}1&\binom{n}{1}&\binom{n}{2}&\binom{n}{3}&\dots&\binom{n}{d+1}\\0&1&\binom{n}{1}&\binom{n}{2}&\dots&\binom{n}{d}\\0&0&1&\binom{n}{1}&\ddots&\vdots\\0&0&0&1&\ddots&\binom{n}{2}\\\vdots&\vdots&\vdots&\ddots&\ddots&\binom{n}{1}\\0&0&0&\dots&0&1\end{bmatrix}$$

Note that for $d = 1$, this is the matrix T_1 we used earlier to generate difference table 2.

Proof. We proceed by induction. The case $n = 1$ holds trivially since $\binom{1}{k} = 0$ for $k > 1$. Assume that

$$\begin{bmatrix}1&1&0&0&\dots&0\\0&1&1&0&\dots&0\\0&0&1&1&\ddots&\vdots\\0&0&0&1&\ddots&0\\\vdots&\vdots&\vdots&\ddots&\ddots&1\\0&0&0&\dots&0&1\end{bmatrix}^m = \begin{bmatrix}1&\binom{m}{1}&\binom{m}{2}&\binom{m}{3}&\dots&\binom{m}{d+1}\\0&1&\binom{m}{1}&\binom{m}{2}&\dots&\binom{m}{d}\\0&0&1&\binom{m}{1}&\ddots&\vdots\\0&0&0&1&\ddots&\binom{m}{2}\\\vdots&\vdots&\vdots&\ddots&\ddots&\binom{m}{1}\\0&0&0&\dots&0&1\end{bmatrix}$$

Then we have that

$$\begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}^{m+1} = \begin{bmatrix} 1 & \binom{m}{1} & \binom{m}{2} & \dots & \binom{m}{d+1} \\ 0 & 1 & \binom{m}{1} & \ddots & \binom{m}{d} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \binom{m}{1} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \binom{m}{0}+\binom{m}{1} & \binom{m}{1}+\binom{m}{2} & \dots & \binom{m}{d}+\binom{m}{d+1} \\ 0 & 1 & \binom{m}{0}+\binom{m}{1} & \ddots & \binom{m}{d-1}+\binom{m}{d} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \binom{m}{0}+\binom{m}{1} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \binom{m+1}{1} & \binom{m+1}{2} & \dots & \binom{m+1}{d+1} \\ 0 & 1 & \binom{m+1}{1} & \ddots & \binom{m+1}{d} \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \binom{m+1}{1} \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

using the recurrence relation $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ for the binomial coefficients. ■

Proposition 2.

$$\sum_{k=1}^{n} k = \binom{n+1}{2}$$

Proof. By Proposition 1 ($d = 1$) and equation(5), we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{n} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & \binom{n}{1} & \binom{n}{2} \\ 0 & 1 & \binom{n}{1} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \binom{n}{1}+\binom{n}{2} \\ 1+\binom{n}{1} \\ 1 \end{bmatrix} = \begin{bmatrix} \binom{n+1}{2} \\ 1+n \\ 1 \end{bmatrix},$$

which implies the proposition. ■

Main results

Next we consider the following sums, where $d \in \mathbb{N}$:

$$S_d(n) = 0^d + 1^d + 2^d + 3^d + \cdots + n^d = \sum_{k=0}^{n} k^d \tag{6}$$

Below is a portion of the difference table of the sequence $(S_d(n))_{n\geq 0}$. We start the sequence conveniently at zero, so that the first difference sequence is precisely the dth powers of consecutive natural numbers.

Difference Table 3

$$
\begin{array}{ccccccccc}
0, & & 1^d, & & 1^d+2^d, & & 1^d+2^d+3^d, & & 1^d+2^d+3^d+4^d, \quad \ldots \\
& 1^d, & & 2^d, & & 3^d, & & 4^d, \quad \ldots & \\
& & 2^d-1^d, & & 3^d-2^d, & & 4^d-3^d, \quad \ldots & & \\
& & & 3^d-2\cdot 2^d+1^d, & & 4^d-2\cdot 3^d+2^d, \quad \ldots & & &
\end{array}
$$

and so on. Since the sum of the dth powers of the first n natural numbers is a polynomial of degree $d+1$ in n, it is well known that $\Delta^{d+1}S_d(n)$ will be constant and $\Delta^{d+2}S_d(n) = 0$ for all n. Actually, in Proposition 4 we show that $\Delta^{d+1}S_d(n) = d!$, which is to be expected.

Proposition 3.

$$\Delta^k S_d(n) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} (n+j)^d \qquad k \geq 1$$

Proof. We proceed by induction on k. For $k = 1$ we have

$$\Delta S_d(n) = \sum_{j=1}^{1} (-1)^{1-j} \binom{0}{j-1} (n+j)^d = (n+1)^d \qquad n \geq 0$$

which is the sequence $1^d, 2^d, 3^d, \ldots$ as expected. Assuming that

$$\Delta^k S_d(n) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} (n+j)^d,$$

we will show that

$$\Delta^{k+1} S_d(n) = \sum_{j=1}^{k+1} (-1)^{k+1-j} \binom{k}{j-1} (n+j)^d.$$

We would like to point out to the reader the following interesting induction step. To the authors, this seems to be an intriguing and perhaps unexpected construction vis-á-vis a classical induction proof.

$$
\begin{aligned}
\Delta^{k+1} S_d(n) &= \Delta^k S_d(n+1) - \Delta^k S_d(n) \\
&= \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} (n+j+1)^d - \sum_{j=1}^{k} (-1)^{k-j} \binom{k-1}{j-1} (n+j)^d \\
&= (-1)^0 \binom{k-1}{k-1} (n+k+1)^d + \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k-1}{j-1} (n+j+1)^d \\
&\qquad - \sum_{j=2}^{k} (-1)^{k-j} \binom{k-1}{j-1} (n+j)^d - (-1)^{k-1} \binom{k-1}{0} (n+1)^d
\end{aligned}
$$

$$= (n+k+1)^d + \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k-1}{j-1} (n+j+1)^d$$

$$+ \sum_{j=2}^{k} (-1)^{k-j+1} \binom{k-1}{j-1} (n+j)^d + (-1)^k (n+1)^d$$

$$= (n+k+1)^d + \sum_{j=2}^{k} (-1)^{k-j+1} \binom{k-1}{j-2} (n+j)^d$$

$$+ \sum_{j=2}^{k} (-1)^{k-j+1} \binom{k-1}{j-1} (n+j)^d + (-1)^k (n+1)^d$$

$$= (n+k+1)^d + \sum_{j=2}^{k} (-1)^{k-j+1} \left[\binom{k-1}{j-2} + \binom{k-1}{j-1} \right] (n+j)^d$$

$$+ (-1)^k (n+1)^d$$

$$= (n+k+1)^d + \sum_{j=2}^{k} (-1)^{k-j+1} \binom{k}{j-1} (n+j)^d + (-1)^k (n+1)^d$$

$$= \sum_{j=1}^{k+1} (-1)^{k-j+1} \binom{k}{j-1} (n+j)^d$$

■

Let n and k be positive integers. The Stirling numbers of the second kind, denoted by $\left\{ {n \atop k} \right\}$, are defined by the recurrence relation

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}, \tag{7}$$

or more explicitly by

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n \tag{8}$$

For $n > 0$, $\left\{ {n \atop 0} \right\} = 0$. For $k > n$, $\left\{ {n \atop k} \right\} = 0$, $\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = \left\{ {0 \atop 0} \right\} = 1$, and $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$.

Combinatorially, the Stirling numbers of the second kind represent the number of ways we can partition n objects into k nonempty subsets. They also appear as coefficients when we convert ordinary powers into falling factorial powers through the formula

$$n^d = \sum_{k=1}^{d} [n]_k \left\{ {d \atop k} \right\} = \sum_{k=1}^{d} \binom{n}{k} k! \left\{ {d \atop k} \right\} \tag{9}$$

where $[n]_k = n(n-1)(n-2)\cdots(n-k+1) = \binom{n}{k} k!$. In (9), we see the reason why Stirling numbers of the second kind and binomial coefficients are present in some of the formulas for sums of powers of natural numbers.

Proposition 4. *The* 0*th diagonal of the difference table of the sequence* $(S_d(n))_{n\geq 0}$ *is given by*

$$\Delta^k S_d(0) = \begin{cases} (k-1)!\left\{ {d+1 \atop k} \right\}, & \text{if } k \geq 1; \\ 0, & \text{if } k = 0. \end{cases}$$

Proof. By Proposition 3,

$$\begin{aligned} \Delta^k S_d(0) &= \sum_{j=1}^{k}(-1)^{k-j}\binom{k-1}{j-1} j^d = \sum_{j=1}^{k}(-1)^{k-j}\left[\binom{k}{j} - \binom{k-1}{j}\right] j^d \\ &= \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} j^d - \sum_{j=1}^{k}(-1)^{k-j}\binom{k-1}{j} j^d \\ &= \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j} j^d + \sum_{j=1}^{k}(-1)^{(k-1)-j}\binom{k-1}{j} j^d \\ &= k!\left\{ {d \atop k} \right\} + \sum_{j=1}^{k-1}(-1)^{(k-1)-j}\binom{k-1}{j} j^d = k!\left\{ {d \atop k} \right\} + (k-1)!\left\{ {d \atop k-1} \right\} \\ &= (k-1)!\left[k\left\{ {d \atop k} \right\} + \left\{ {d \atop k-1} \right\}\right] = (k-1)!\left\{ {d+1 \atop k} \right\} \end{aligned}$$

■

We note that in the above proof both (7) and (8) were used, along with the recurrence relation of the binomial coefficients. As expected based on Proposition 4,

$$\Delta^1 S_d(0) = 0!\left\{ {d+1 \atop 1} \right\} = 1, \quad \Delta^2 S_d(0) = 1!\left\{ {d+1 \atop 2} \right\} = 2^d - 1$$

$$\Delta^{d+1} S_d(0) = d!\left\{ {d+1 \atop d+1} \right\} = d!$$

In Proposition 4, we essentially establish initial conditions for all difference sequences at zero, similar to computing $f^{(n)}(0)$ to express an analytic function f as a Taylor Series about $x = 0$.

We now consider the following successive multiplications of the 0th diagonal (vector) $\Delta^k S_d(0)$ $(0 \leq k \leq d+1)$, of the sequence $(S_d(n))_{n\geq 0}$ by the matrix T_d.

$$T_d \Delta^k S_d(0) = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2^d - 1 \\ 3^d - 2\cdot 2^d + 1^d \\ \vdots \\ d! \end{bmatrix} = \begin{bmatrix} 1^d \\ 2^d \\ 3^d - 2^d \\ 4^d - 2\cdot 3^d + 2^d \\ \vdots \\ d! \end{bmatrix}$$

$$T_d^2 \begin{bmatrix} 0 \\ 1 \\ 2^d - 1 \\ 3^d - 2\cdot 2^d + 1^d \\ \vdots \\ d! \end{bmatrix} = T_d \begin{bmatrix} 1^d \\ 2^d \\ 3^d - 2^d \\ 4^d - 2\cdot 3^d + 2^d \\ \vdots \\ d! \end{bmatrix} = \begin{bmatrix} 1^d + 2^d \\ 3^d \\ 4^d - 3^d \\ 5^d - 2\cdot 4^d + 3^d \\ \vdots \\ d! \end{bmatrix} \quad \cdots$$

This enables navigation through the entire differrence table and is the key to finding closed forms for all the difference sequences in difference Table 3.

Theorem 5. *Let n and $d \in \mathbb{N}$. Then*

$$\sum_{k=1}^{n} k^d = 1^d + 2^d + 3^d + \cdots + n^d = \sum_{k=1}^{d+1} \binom{n}{k} (k-1)! \left\{ {d+1 \atop k} \right\}$$

Proof. Since

$$T_d^n \Delta^k S_d(0) = \begin{bmatrix} 1^d + 2^d + 3^d + \cdots + n^d \\ (1+n)^d \\ (2+n)^d - (1+n)^d \\ \vdots \\ d! \end{bmatrix},$$

by Propositions 1 and 4,

$$\begin{bmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{d+1} \\ 0 & 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{d} \\ 0 & 0 & 1 & \binom{n}{1} & \ddots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & \binom{n}{2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \binom{n}{1} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0!\left\{ {d+1 \atop 1} \right\} \\ 1!\left\{ {d+1 \atop 2} \right\} \\ 2!\left\{ {d+1 \atop 3} \right\} \\ \vdots \\ d!\left\{ {d+1 \atop d+1} \right\} \end{bmatrix} = \begin{bmatrix} 1^d + 2^+ \cdots + n^d \\ (1+n)^d \\ (2+n)^d - (1+n)^d \\ \vdots \\ d! \end{bmatrix}. \tag{10}$$

The first product in (10) yields

$$\begin{aligned} \sum_{k=1}^{n} k^d &= \binom{n}{1} 0! \left\{ {d+1 \atop 1} \right\} + \binom{n}{2} 1! \left\{ {d+1 \atop 2} \right\} + \cdots + \binom{n}{d+1} d! \left\{ {d+1 \atop d+1} \right\} \\ &= \sum_{k=1}^{d+1} \binom{n}{k} (k-1)! \left\{ {d+1 \atop k} \right\} \end{aligned}$$

■

Theorem 5 essentially proves Newton's Theorem: If f is a polynomial of degree n in the variable x, then

$$f(x) = \sum_{k=0}^{n} \frac{\Delta^k f(0)}{k!} [x]_k = \sum_{k=0}^{n} \Delta^k f(0) \binom{x}{k} \tag{11}$$

The reader may have recognized in (11) the discrete analogue of a Taylor series. Note that the ordinary powers in the Taylor series have been replaced by falling factorial powers, and the nth order derivatives have been replaced by the nth order finite differences. The following are a few of the corollaries of Theorem 5.

Corollary 6.

$$\sum_{k=1}^{d+1}\binom{n}{k}(k-1)!\left\{{d+1 \atop k}\right\} = \sum_{k=1}^{d}\binom{n+1}{k+1}k!\left\{{d \atop k}\right\}$$

Proof.

$$\begin{aligned}\sum_{k=1}^{d+1}\binom{n}{k}(k-1)!\left\{{d+1 \atop k}\right\} &= \sum_{k=1}^{d+1}\binom{n}{k}(k-1)!\left[\left\{{d \atop k-1}\right\} + k\left\{{d \atop k}\right\}\right] \\ &= \sum_{k=1}^{d+1}\binom{n}{k}(k-1)!\left\{{d \atop k-1}\right\} + \sum_{k=1}^{d+1}\binom{n}{k}k!\left\{{d \atop k}\right\} \\ &= \sum_{k=0}^{d}\binom{n}{k+1}k!\left\{{d \atop k}\right\} + \sum_{k=1}^{d+1}\binom{n}{k}k!\left\{{d \atop k}\right\} \\ &= \sum_{k=1}^{d}\left[\binom{n}{k+1} + \binom{n}{k}\right]k!\left\{{d \atop k}\right\} \\ &= \sum_{k=1}^{d}\binom{n+1}{k+1}k!\left\{{d \atop k}\right\}\end{aligned}$$

■

Corollary 7.

$$\Delta^d S_d(n) = \frac{(d+1)!}{2} + d!n$$

Proof. By computing the second product from the bottom in (10), we get

$$\begin{aligned}\Delta^d S_d(n) &= \left\{{d+1 \atop d}\right\}(d-1)! + \binom{n}{1}\left\{{d+1 \atop d+1}\right\}d! \\ &= \binom{d+1}{2}(d-1)! + d!n = \frac{(d+1)!}{2} + d!n.\end{aligned}$$

Note that $\left\{{d+1 \atop d}\right\} = \binom{d+1}{2}$, since there are $\binom{d+1}{2}$ ways to partition a set of $d+1$ elements into d distinct subsets. ■

Corollary 8.

$$(1+n)^d = \sum_{k=0}^{d}\binom{n}{k}\left\{{d+1 \atop k+1}\right\}k! = \sum_{k=0}^{d}[n]_k\left\{{d+1 \atop k+1}\right\}$$

We leave the proof for the reader, which is straightforward based on (10). Corollary 8 captures a special case of the binomial theorem where the role of the binomial coefficients is being played by the Stirling numbers of the second kind, and regular powers are being replaced by falling factorial powers.

Final remarks

We proved the formula for the sums of powers of consecutive natural numbers in the form given by Newton's theorem. A proof of the same formula by induction can be found in Butzer, Kilbas and Trujillo [4]. Our perhaps somewhat atypical technique has generated additional results about difference sequences, some of which are corollaries of Theorem 5.

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Summary. We prove a known formula for the sums of powers of consecutive natural numbers by employing matrices to exploit sequences and their corresponding difference tables. Our approach leads to a formula for sums of powers that involves the Stirling numbers of the second kind, as well as additional results about difference sequences.

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Making Ellipses by Folding Disks

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We are all familiar with the following method for constructing an ellipse: Fix two points, f_1 and f_2, called *the foci* of the ellipse. Put a thumbtack at f_1 and another at f_2, and tie a piece of string of fixed length (greater than the distance between f_1 and f_2) to each of the foci. Put a pencil in the string and move it so that the string is taut. Move the pencil around the tacks until you are back where you started. The curve you trace out is an ellipse, and the two foci lie on the *major axis*. The *center* of the ellipse is the point halfway between the foci, and the *minor axis* is the line segment perpendicular to the major axis, passing through the center and having endpoints on the ellipse. Thus, an ellipse is the locus of points P such that the sum of the distance from P to f_1 and the distance from P to f_2 is the same for all points P. If you move the foci closer together, the ellipse becomes "more circular" (less eccentric) and if the foci are the same, you have traced out a circle.

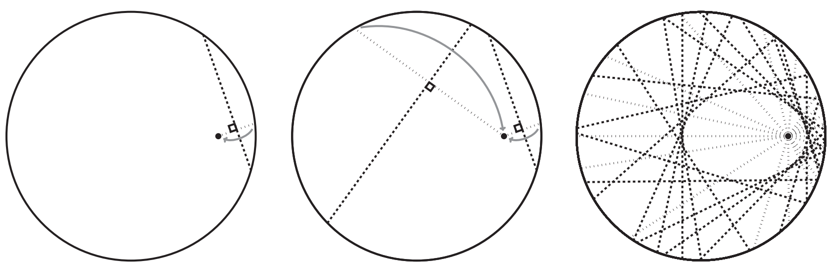

Figure 1 One fold, two folds, many folds.

There are many other ways to trace an ellipse. The method we study in this paper was popularized by Martin Gardner in his article "The Ellipse" [**3**, Chapter 15]. Martin Gardner is perhaps best known for his "Mathematical Games" columns in *Scientific American* (as well as the books compiling them)—some of his many works promoting and exploring "recreational mathematics." Gardner's method is as follows: "Cut a large circle from a sheet of paper. Make a spot somewhere inside the circle, but not at the center, then fold the circle so that its circumference falls on the spot" (see the picture on the left of Figure 1). "Unfold, then fold again, using a different point on the circumference, and keep repeating this until the paper has been creased many times in all directions." The picture on the right of Figure 1 shows many folds. It appears that these lines are tangent to an ellipse, and it also appears that the foci are at the center c of the circle and the point d that we chose. Is this an ellipse? If so, are c and d the foci?

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MSC: 30D05, 30J10

If this really is an ellipse with foci at the points c and d, then the distance from a point p on the ellipse to d, denoted $|\overline{pd}|$, plus the distance from p to c must be a fixed constant independent of the point on the ellipse that we consider. So pick a point a on the circle and fold so that a lies on top of d. Let b and m be the points where the fold (the dotted line) intersects $\overline{ac}$ and $\overline{ad}$, respectively; see Figure 2.

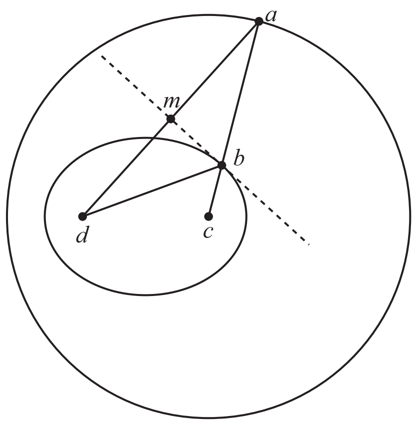

Figure 2 Start with an arbitrary point a on the circle.

We obtained m by folding, so $|\overline{am}| = |\overline{md}|$ and $\angle amb$ and $\angle dmb$ are right angles. Thus the two triangles, Δamb and Δdmb are congruent and $|\overline{ab}| = |\overline{bd}|$. Putting this together we have

$$|\overline{bd}| + |\overline{bc}| = |\overline{ab}| + |\overline{bc}|.$$

But b lies on the line $\overline{ac}$, so $|\overline{ac}| = |\overline{ab}| + |\overline{bc}|$. Now $\overline{ac}$ is the radius of the circle we started with, so the sum of the distances of b to the foci is a fixed constant and that constant is the radius r of the circle. Finally, if you pick a point $x \neq b$ on the line passing through b and m, the same argument shows that

$$|\overline{dx}| + |\overline{cx}| = |\overline{ax}| + |\overline{cx}|.$$

But a, x, and c are not collinear, so $|\overline{dx}| + |\overline{cx}| > r$, and b is the only point on the line that touches the ellipse. That is, the fold is tangent to the ellipse at b. We call an ellipse produced by this method a *Gardner ellipse*.

Modify this construction as follows: Form the first fold. Then, rather than choosing an arbitrary point to create the second fold, proceed in such a way as to start the second

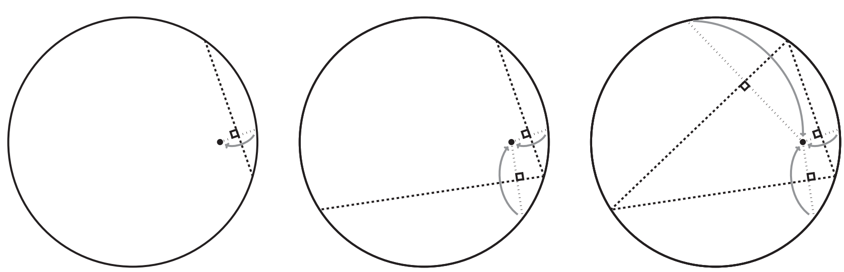

Figure 3 A sequence of three folds closes up.

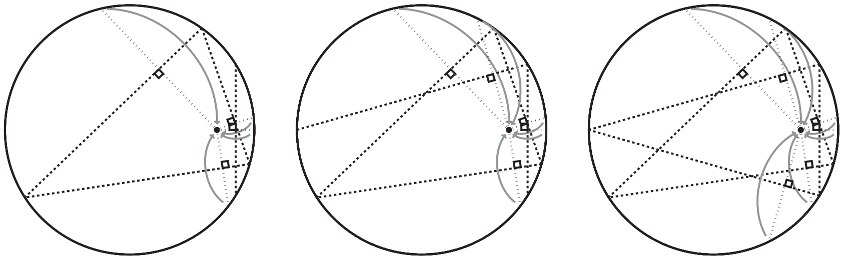

Figure 4 A second sequence of three folds closes up.

fold where the first ended; see Figure 3. Then start the third fold where the second ended. We call this *sequential folding*.*

Looking at Figure 3, we see that the line segments form a triangle! It is certainly possible that the points we chose were special in some way, so let us try again. In Figure 4, we see that this triangle also closes up.

In fact, every sequential folding will produce a triangle with vertices on the circle that circumscribes an ellipse (in fact, the Gardner ellipse) and the goal of this paper is to explain why that is so.

Poncelet's theorem

Jean-Victor Poncelet was born in France in 1788. He graduated from the École Polytechnique in 1810 and later joined Napoleon's army as it approached Russia. According to some sources* "Poncelet was left for dead on the battlefield following the Battle of Krasnoi." He was forced to march to prison in Saratov, where he was held from March of 1813 to June of 1814. It was during this time that he wrote the *Saratov notebook*, which later become his *Traité des propriétés projectives des figures* [**5**]. He proved the following theorem, now bearing his name.

Theorem 1 (Poncelet's theorem)**.** *Let C and D be two ellipses with D contained inside C. If, for $n \geq 3$, it is possible to find an n-sided polygon that is simultaneously inscribed in C (that is, all of its vertices lie on C) and circumscribed around D (that is, all of its edges are tangent to D), then each point of C is a vertex of such a polygon.*

Poncelet's version of the theorem was more general than this, but we need only this form. For more recent articles on this theorem, we suggest Daepp, Gorkin, Shaffer, and Voss [**2**], Halbeisen and Hungerbühler [**4**], and Simonič [**6**].

It appears that if we use sequential folding, the folds form a triangle. So the triangle circumscribes an ellipse (playing the role of D in Poncelet's theorem) and has all its vertices on the circle (playing the role of C). If one sequence of folds closes up into a triangle, Poncelet's theorem tells us that every sequence of folds must also do so. So let us turn to the proof that one triangle circumscribes the ellipse, and then Poncelet's theorem will do the rest of the work for us.

Theorem 2. *Use sequential folding to create an ellipse E. Then three sequential folds form a triangle circumscribing E.*

The proof is based on Figures 5 and 6.

*Sequential folding with physical paper can be tricky. Try holding the fixed end of the fold and "pivoting" the paper until the circle touches the point d.

*history.mcs.st-and.ac.uk/Biographies/Poncelet.html

According to Wikipedia, his biographer gives a slightly different account.

Proof. Rotating and scaling, if necessary, we may assume the center of the circle is $0 = (0, 0)$, the radius is 1, and the second point that we pick, denoted by a, is on the positive x-axis. (Throughout we use a to refer to both the scalar distance from the origin as well as to the point $(a, 0)$, as necessary.) We wish to show that the Gardner ellipse is circumscribed by a triangle with vertices on the unit circle, so we recall that our previous work shows that the equation of the ellipse E is $|\overline{0z}| + |\overline{az}| = 1$. By Poncelet's theorem, we have only to show that one triangle circumscribes E and is inscribed in the unit circle. We pick a triangle that takes advantage of the symmetry of the ellipse and circle.

We start our sequence by folding the point $(1, 0)$ onto the point a, and observe that the point $(\frac{1+a}{2}, 0)$ is on E. Since E is symmetric across the x-axis, the fold is vertical, and it intersects the circle at the two points

$$\left(\frac{1+a}{2}, \sqrt{1-\left(\frac{1+a}{2}\right)^2}\right) \quad \text{and} \quad \left(\frac{1+a}{2}, -\sqrt{1-\left(\frac{1+a}{2}\right)^2}\right).$$

By symmetry, the common endpoint for the two remaining sides must be $(-1, 0)$, as shown in Figure 5.

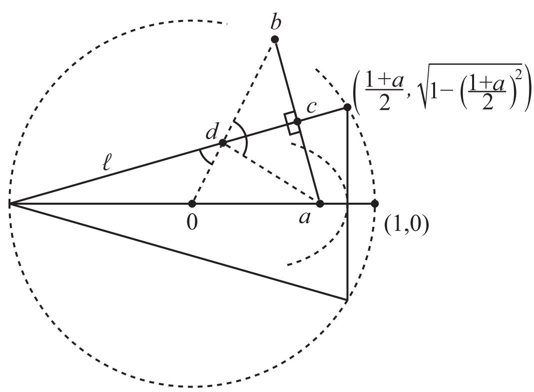

Figure 5 The circle and the point b. Since we do not yet know that b lies on the circle, we have not drawn full circles in our figures.

When we draw the triangle, we know only that the first side is a fold, and we aim to show that the other two sides are, in fact, the next sequential folds. We show this by proving that they are tangent to the Gardner ellipse E. Let ℓ be the line joining

$$\left(\frac{1+a}{2}, \sqrt{1-\left(\frac{1+a}{2}\right)^2}\right) \quad \text{and} \quad (-1, 0),$$

and let d be the point on ℓ at which the acute angle between $\overline{0d}$ and ℓ equals the acute angle between $\overline{ad}$ and ℓ (see Figure 5). We note that the point d exists and is unique. (We see this by considering lines perpendicular to ℓ through each of 0 and a, and looking at the relevant angles if d were at each intersection between those lines and ℓ.) Now, it is clear by the reflection property of ellipses that ℓ is tangent at d to some ellipse with foci 0 and a, and we prove that it is in fact tangent to E.

Draw the line passing through a that is perpendicular to ℓ. We denote the point of intersection of this line with ℓ by c (see Figure 5). This line passes through a and c, so we denote it by $\overleftrightarrow{ac}$. Extend the line segment $\overline{0d}$ until it meets $\overleftrightarrow{ac}$ at a point b (see Figure 5). Note also that the lines $\overleftrightarrow{0d}$ and $\overleftrightarrow{ac}$ intersect above ℓ, since $\angle cd0$ is obtuse by the construction of d and $\overline{ac}$ is perpendicular to ℓ. Once we show b is on the circle and the length of $\overline{bc}$ is equal to the length of $\overline{ca}$, we have shown that ℓ is a fold.

By the choice of d and the fact that the vertex angles at d are equal, we see that $\triangle bdc$ and $\triangle adc$ have all angles equal. But they also share the common side $\overline{dc}$, so they are congruent. In particular, $|\overline{bc}| = |\overline{ca}| := x$ and $|\overline{db}| = |\overline{da}|$. Now if we show that $|\overline{0b}| = 1$, then b lies on the circle, and

$$1 = |\overline{0b}| = |\overline{0d}| + |\overline{db}| = |\overline{0d}| + |\overline{da}|.$$

So we will have shown that ℓ is a fold and d lies on the ellipse E, or $|\overline{0z}| + |\overline{az}| = 1$. (The fact that ℓ is a line of tangency follows from our choice of d as the point where the reflection property holds.)

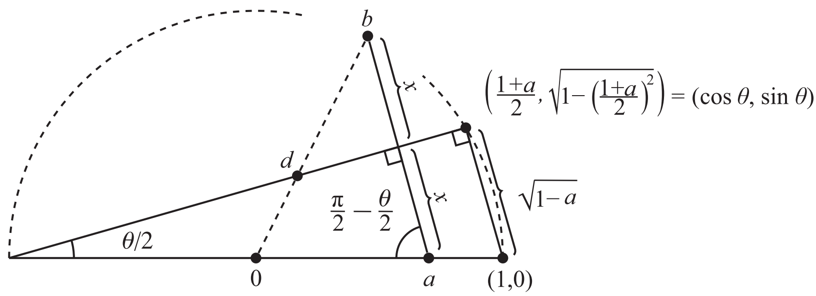

Figure 6 Two similar right triangles.

To show that $|\overline{0b}| = 1$, consider Figure 6, and note that the two right triangles with hypotenuses on the diameter of the circle are similar. Thus,

$$\frac{x}{\sqrt{1-a}} = \frac{1+a}{2}.$$

So $2x = (1+a)\sqrt{1-a}$. Further,

$$\sin\left(\frac{\theta}{2}\right) = \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\sqrt{1-a}}{2}.$$

An application of the law of cosines tells us that

$$\begin{aligned}
|0b|^2 &= a^2 + (2x)^2 - 2a(2x)\cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right) \\
&= a^2 + (1+a)^2(1-a) - 2a(1+a)\sqrt{1-a} \cdot \frac{\sqrt{1-a}}{2} \\
&= a^2 + (1+a)^2(1-a) - a(1+a)(1-a) \\
&= a^2 + (1-a^2)\left((1+a) - a\right) = 1.
\end{aligned}$$

Symmetry shows that the final side of the triangle is also a fold and tangent to E. Finally, Poncelet's theorem completes the proof. ■

Remark. There is another proof of this fact based on functions known as Blaschke products. Let a and b be complex numbers of modulus less than one and let

$$B(z) = z\left(\frac{z-a}{1-\overline{a}z}\right)\left(\frac{z-b}{1-\overline{b}z}\right).$$

It is well known that B maps the unit circle to itself and that there are, for each λ in the unit circle, three distinct solutions in the circle to $B(z) = \lambda$. It can be shown [**1**] that the triangle with these three solutions as vertices circumscribes the ellipse E_1 with equation

$$|z-a| + |z-b| = |1-\overline{a}b|.$$

Thus, E_1 is a Poncelet ellipse. Since it is inscribed in one triangle with vertices on the unit circle, every point of the unit circle is the vertex of such a circumscribing triangle. When we considered Gardner's ellipse, we saw that it had, assuming the center of the circle is at 0 and the radius is 1, equation $|z| + |z-d| = 1$. Therefore, Gardner's ellipse is the ellipse associated with the Blaschke product with $a = 0$ and $b = d$. As such, it must be a Poncelet ellipse. For more information about this, see the recent book by Daepp, Gorkin, Shafffer, and Voss [**2**]. Though this proof of Theorem 2 is shorter, it is not as elementary.

Acknowledgments Finally, we would also like to thank the referees for their very helpful comments. This work was partially supported by a grant from the Simons Foundation (♯243653 to Pamela Gorkin).

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Summary. In an article entitled *The Ellipse*, Martin Gardner explains a simple way to "fold" an ellipse inside a circle. It is surprising that an ellipse appears through folding, but there are more surprises in store: For every point on the circle there is a triangle with that point as one of the vertices that both circumscribes the ellipse and is inscribed in the circle.

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A Prime Example of the Strong Law of Small Numbers

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Richard Guy gives 115 "examples of patterns that *seem* to appear when we look at several small values of n" [**1**, **2**]. He uses these examples to posit his *Strong Law of Small Numbers*—"There aren't enough small numbers to meet the many demands made of them."

He invites readers to let him know "if I've missed out [on] your favorite example." We have a good, new one.

Our first sequence is:

$$\text{Round}(n\ln(n)) = 0, 1, 3, 6, 8, 11, 14, 17, 20, 23, 26, 30, 33, \ldots$$

This sequence is important because the prime number theorem is equivalent to the statement that the n^{th} prime is asymptotic to Round$(n\log(n))$ [**4**, pp. 11–12].

Our "matching" sequence arises from landscape ecology, where ecologists studying erosion use *landscape indices* to measure landscape compactness.

The widely used *aggregation index* measures how many pixels representing land are edge-connected to others in a satellite image. Geometrically, we consider regions that are *polyplets*—connected collections of squares each of which is attached to another by a shared edge or shared vertex. *Cardinal adjacencies* are counted one each for shared edge connections in the polyplet. *Intercardinal adjacencies* are counted one each for shared edges *and* shared vertices. The situation is illustrated in Figure 1.

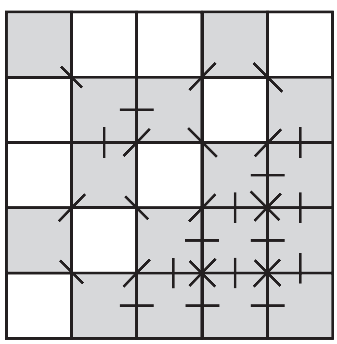

Figure 1 Dark areas represent land and white areas represent water. This is a 16-cell polyplet. As shown with horizontal and vertical tick marks, there are $\mu = 14$ cardinal adjacencies. Including diagonal tick marks, there are $M = 31$ intercardinal adjacencies.

It is valuable to know what configurations maximize the number of cardinal adjacencies. Harary and Harborth proved that among all polyplets constructed from n

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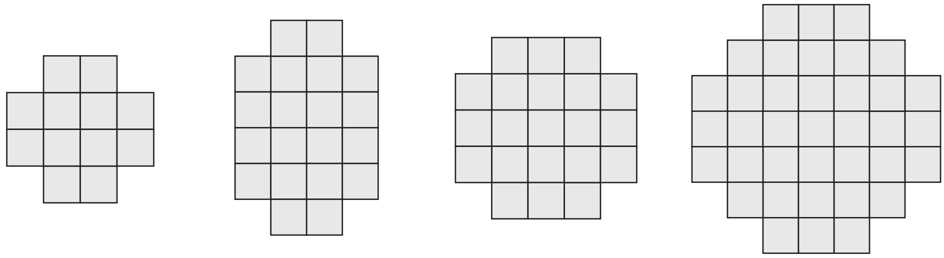

Figure 2 Regular octagon with $M_O(12) = 30$; stretched octagon with $M_O(16) = 43$; small corners octagon with $M_O(21) = 60$; regular octagon with $M_O(37) = 116$.

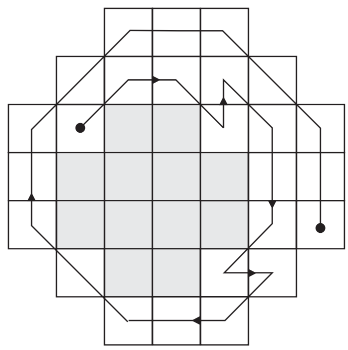

Figure 3 Octagon spiral for building archetypes.

squares, the optimal number of cardinal adjacencies, that is, the number of shared edges, is given by the sequence

$$\mu(n) = 0, 1, 2, 4, 5, 7, 8, 10, 12, 13, 15, 17, 18, 20, 22, \ldots = 2n - \lceil\sqrt{4n}\rceil.$$

Optimal configurations are formed by a "square spiral"—adding tiles one after another along a side of a k by k square to form a k by $k+1$ rectangle, then adding tiles one after another along the longer edge to form a $k+1$ by $k+1$ square, and then continuing in this same fashion [3].

In coastal regions like the Louisiana Gulf Coast, where land/water boundaries are of high density, land cells that share vertices, but not necessarily entire edges, will also be of value in stemming erosion. In situations like this, measuring the number of intercardinal adjacencies, rather than cardinal adjacencies, is most useful.

At the request of a landscape ecologist doing this type of quantitative analysis, we were asked to consider the following problem:

Problem. Among all polyplets constructed from n squares, what is the largest number, $M(n)$, of intercardinal adjacencies?

After finding maximal configurations for the smaller polyplets, we consulted with the Online Encyclopedia of Integer Sequences [5]. And there it is, what seems to be the answer to our problem:

$$M(n) = \text{Round}(n\ln(n)).$$

Indeed, for $n = 0$ to $n = 13$ we have:

$$M(n) = 0, 1, 3, 6, 8, 11, 14, 17, 20, 23, 26, 30, 33 = \text{Round}(n\ln(n)).$$

However,

$$36 = M(14) \neq \text{Round}(14 \ln(14)) = 37.$$

And, in fact, $M(n) \neq \text{Round}(n \ln(n))$ for all $14 \leq n \leq 10^7$.

Nonetheless, $M(n)$ agrees with $\text{Round}(n \ln(n))$ longer than any other sequence on OEIS, and is therefore worthy of Guy's invitation to share.

What about the solution to our problem: What is $M(n)$? Data and intuition suggest that the optimal configurations arise from "octagon spirals." Figure 2 shows how we move along the geometric archetypes: regular octagon → stretched octagon → small corners octagon → regular octagon. Notice that this construction yields a spiral that is staggered, as shown in Figure 3.

We have proved that the number of intercardinal adjacencies in each stage in the family of octagon spirals, as shown in Figure 2, is given by $M_O(n) = 4n - \lceil\sqrt{28n - 12}\rceil$. So we conjecture that $M(n) = M_O(n)$. We have proved that $M_O(n) \leq M(n) \leq 2\mu(n)$. The upper bound appears never to be sharp for $n > 1$ [6]. The relative difference between these upper and lower bounds goes to zero and is less than 3% for at least $145 \leq n \leq 10^7$, so our result is sufficient for use in landscape ecology. We hope that future work will result in a full solution to our problem—a proof that $M_O(n) = M(n)$ for all n.

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Summary. We give a new illustration of Richard Guy's "Strong Law of Small Numbers." This illustration uses a new sequence which agrees with *Round*$(n \ln(n))$ through the first 13 terms. Our sequence is the answer to a new geometric combinatorics problem—one with no expected connections to the asymptotic approximation to the primes.

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JULIAN F. FLERON is a professor of mathematics at Westfield State University, where he has taught for the last 26 years. He has deeply enjoyed supervising a great many student research projects in many areas. He was principle investigator on the Discovering the Art of Mathematics project (http://artofmathematics.org), which provides rich, inquiry-based investigations for mathematics for liberal arts students. Exploring mathematics using this material in a general education course, the first author began an unexpected journey to a career in mathematics. This work is a small part of a years-long collaboration between the authors.

Counting Spaces

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Consider a permutation of a finite set. Recall that a *swap* is a cycle of length two, and a permutation is *odd* or *even* if it can be written as the product of an odd or even number of swaps, respectively. With this in mind, take a moment to consider whether an arbitrary permutation, say

$$(1\ 4\ 5\ 6)\ (4\ 3\ 1)\ (9\ 5)\ (2)\,,$$

is odd or even. This probably took a little longer than counting to six, but it doesn't have to.

To see what we mean, consider that when we write a permutation as the product of cycles, what we actually write on the page are elements of our set surrounded by parentheses. Within a cycle, we separate elements of the set by leaving a gap. Let us call each of these gaps *spaces*. This terminology allows us to state the following.

Proposition. *A permutation is odd if it can be written with an odd number of spaces and even if it can be written with an even number of spaces.*

Thus

$$(1\ 4\ 5\ 6)\ (4\ 3\ 1)\ (9\ 5)\ (2)$$

is even because it is written with six spaces. To prove the proposition, we use this lemma:

Lemma 1. *A permutation can be written with m spaces if and only if it can be written as the product of m swaps.*

Proof. First notice that the product of m swaps

$$(a_1\ b_1)\ (a_2\ b_2) \cdots (a_m\ b_m)$$

is already written with m spaces. For the opposite implication, consider that a single cycle written with i spaces, say $(c_0\ c_1\ \cdots\ c_i)$, can be written as the product of i swaps as

$$(c_0\ c_1)\ (c_0\ c_2) \cdots (c_0\ c_i)\,.$$

Thus an arbitrary permutation written with m spaces can be written as the product of m swaps. ∎

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MSC code: 20B30 (Symmetric groups)

For example,

$$(1\ 4\ 5\ 6)\ (4\ 3\ 1)\ (9\ 5)\ (2) = (1\ 4)\ (1\ 5)\ (1\ 6)\ (4\ 3)\ (4\ 1)\ (9\ 5)\,.$$

We can also count spaces to prove this well-known theorem.

Theorem 1. *No permutation can be both odd and even.*

Another useful result comes from counting spaces in a permutation that is written as the product of disjoint cycles.

Theorem 2. *If a permutation σ can be written as a product of disjoint cycles using n spaces, then n is the smallest number of swaps whose product is σ.*

We prove both theorems together. To begin, let us show that when writing a permutation as the product of disjoint cycles, the number of spaces is unique for the permutation. To see this, consider the differences in two such ways of writing a permutation. Although the same cycles of length at least two appear, each cycle might be written starting with a different element. For example, (1 7 6) can be written as (6 1 7). Also, trivial cycles, such as (8), may appear or be omitted. Finally, the cycles can be written in any order. Since these differences do not affect the number of spaces, we can unambiguously refer to the number of spaces when a permutation is written as the product of disjoint cycles. Next, let us look at how this number changes when we multiply by a swap.

Lemma 2. *Take a permutation σ' and a swap $(a\ b)$. Define $\sigma = \sigma'\,(a\ b)$. If we write σ' and σ each as the product of disjoint cycles, then the number of spaces for σ is one more or one less than the number of spaces for σ'.*

Proof. Write σ' as the product of disjoint cycles, where we include the trivial cycles (a) or (b) in case σ' fixes a or b. If there are cycles that do not contain a or b, let τ be their product.

We distinguish two possibilities. Either a and b are in different cycles, say

$$(a\ c_1\ c_2\ \cdots) \qquad \text{and} \qquad (b\ d_1\ d_2\ \cdots)\,,$$

or they are in the same cycle, say

$$(a\ c_1\ c_2\ \cdots\ b\ d_1\ d_2\ \cdots)\,.$$

In either case, we allow the possibility that there are no elements c_i or no elements d_i. Since a, b, the c_i, and the d_i are all distinct, we have that

$$(a\ c_1\ c_2\ \cdots)\,(b\ d_1\ d_2\ \cdots)\,(a\ b) = (a\ c_1\ c_2\ \cdots\ b\ d_1\ d_2\ \cdots)$$

and

$$(a\ c_1\ c_2\ \cdots\ b\ d_1\ d_2\ \cdots)\,(a\ b) = (a\ c_1\ c_2\ \cdots)\,(b\ d_1\ d_2\ \cdots)\,.$$

We see, then, that multiplication by the swap $(a\ b)$ either replaces parentheses)(by a space or vice versa. If there are no other cycles in σ', then one of the expressions on the right gives σ as the product of disjoint cycles. Otherwise we multiply by τ to get this form.

Therefore, when we write σ as the product of disjoint cycles, the number of spaces goes up by one if a and b are in different cycles of σ', and it goes down by one if a and b are in the same cycle of σ'. ■

Next, we claim the following. Let n be the number of spaces when we write an arbitrary permutation σ as the product of disjoint cycles. If σ can be written as the product of m swaps, then $m \geq n$ and $m - n$ is even.

To prove our claim, we induct on m. For $m = 0$, σ is the identity permutation and $n = 0$. For $m = 1$, σ is a single swap and $n = 1$. For $m \geq 2$, write

$$\sigma = (a_1\ b_1) \cdots (a_m\ b_m)\,,$$

and assume that our claim holds for $m - 1$. Let

$$\sigma' = (a_1\ b_1) \cdots (a_{m-1}\ b_{m-1})$$

and let n' be the number of spaces when σ' is written as the product of disjoint cycles. By our inductive hypothesis, $m - 1 \geq n'$ and $(m - 1) - n'$ is even. Lemma 2 tells us $n = n' \pm 1$, so

$$n = n' \pm 1 \leq n' + 1 \leq m.$$

Furthermore,

$$m - n = m - \left(n' \pm 1\right),$$

which is even since $(m - 1) - n'$ is even. Therefore, for all m, we get $m \geq n$ and $m - n$ is even.

For Theorem 1, suppose a permutation σ is written as the product of both m_1 and m_2 swaps. Let n be the number of spaces when we write σ as the product of disjoint cycles. By our claim, $m_1 - n$ and $m_2 - n$ are both even. Therefore,

$$(m_1 - n) - (m_2 - n) = m_1 - m_2$$

is even, which means m_1 and m_2 are both odd or both even.

For Theorem 2, if m_0 is the smallest number of swaps whose product is σ, then $m_0 \geq n$ by the claim and $m_0 \leq n$ by Lemma 1, so $m_0 = n$.

Our proof of Theorem 1, since it ultimately relies on Lemma 2, is essentially equivalent to a proof attributed to David M. Bloom [**1**, Theorem 9.15].

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Summary. When writing down a permutation as the product of cycles, we can count the number of spaces used. This tells us the smallest number of swaps whose product is the permutation, and also whether the permutation is odd or even. It also gives a simple proof that no permutation is both odd and even.

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John Edmark: Art in Motion

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John Edmark teaches in the Design Program at Stanford University. He is the inventor of the Helicone, an interactive kinetic toy, and Blooms, a new type of sculpture that animates when spun under a strobe light. His geometry-based artwork has been featured in the New York Times, Core77, and Colossal. NPR's Science Friday produced a video about him that has received more than 40 million views.

Q: *Most of your art involves motion. What is it about motion that appeals to you?*

JE: The use of motion seems to afford more opportunities for surprising or magical things to happen than would be the case with a static work.

Q: *That really seems to characterize your most recent work with Blooms. Can you describe the techniques that you use to create the visual effects that you see in Blooms?*

JE: Blooms belong to a family of animation devices called zoetropes. In a standard zoetrope, you will have a sequence of images or sculptures rotating in a ring, each of which is slightly different from the previous one. Through either the use of a strobe light or a slit, the device makes it so that the eye sees one image at a time in sequence, creating the illusion of movement, very much like what happens in a motion picture. Blooms operate on a similar principle, with one very significant difference: in a standard zoetrope, sequential frames in time are adjacent in space, but in a Bloom, sequential frames in time are *not* adjacent in space; they are very distant from each other, specifically 137.5 degrees around the central axis from each other. That angle may sound kind of arbitrary until you know that it is actually the angular equivalent of the "golden mean." Because of this relationship, it is called the "golden angle."

Figure 1 Blooms. See them animated at http://www.johnedmark.com/#/phi/.

Q: *The Fibonacci numbers feature in several of your works, and they're really closely related to the golden ratio.*

JE: Yes. In my work where Fibonacci numbers manifest themselves through the number of spirals appearing, they are actually just a symptom of a process. In the case of

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MSC: Primary 01A70

my spiral puzzles, I do explicitly choose certain Fibonacci numbers of spirals to show. However, before I do that, I use the golden angle to lay out a matrix of dots, and then I make decisions about which dots to connect, thereby determining which Fibonacci numbers will appear. Any array of points created using the golden angle imply numerous spirals to choose from, all of which are Fibonacci numbers, but in any given array, only two of the Fibonacci numbers of spirals are going to be most obvious. Others will be less obvious, but they're all there. So, the golden angle is the foundation for building these pieces—both the puzzles and the Blooms—and the Fibonacci numbers are a symptom of that process. In fact, in many of my Bloom designs, I do not explicitly create Fibonacci spirals, but we perceive spirals because our brain's vision processing system, upon being presented with several closely spaced and aligned elements, tends to interpret them as a line. This also happens when we look at golden-angle based plant forms. Artichokes and pineapples and pinecones and sunflowers are not programmed to make specific Fibonacci spirals. Instead, they are repeatedly producing a new leaf, bract, or scale 137.5 degrees around the stem from the previous one. Differences in the amount of time between the emergence of each new element will determine which of the spirals become most prominent.

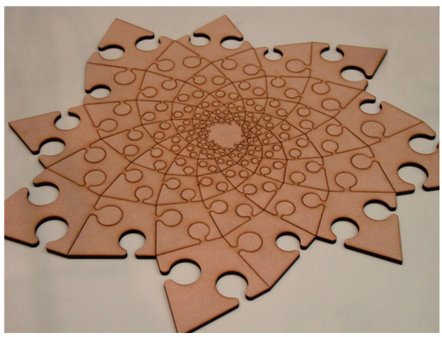

Figure 2 Spiral puzzle.

Q: *So your artistic process is an analogy for what happens naturally.*

JE: Yes! People often ask: how did I come up with Blooms? Blooms are the result of a 10-year-long process of studying and playing with golden angle-based spiral geometries. It was a long incremental process of research and discovery, of surprises and exploiting those surprises. Perhaps surprisingly, I did not invent Blooms by looking at an artichoke or a pineapple and realizing "Oh! If I spin one of these, it will animate!" Instead, I was looking to nature as a source of inspiration for exploring geometry, and eventually came to realize that these golden-angle-based geometries had the potential to animate.

Q: *What kinds of mathematics do you use in your work?*

JE: The math I use is standard geometry, trigonometry, and algebra. It's all high school math. These are incredibly valuable, powerful tools.

Q: *Several of your pieces—like your lollipopters—feel playful. I'm really interested in how play informs mathematical research and artistic processes. Would you say that play is a big part of your artistic process?*

JE: There is definitely play involved. I play with things and learn things from playing. When I make something, I'll have a hypothesis about what it's going to do. Sometimes it does what I think it's going to do, sometimes it doesn't. It's pretty frequent that it

Figure 3 *Helicone*, laser-cut plywood, brass, beechwood, 2008. An interactive kinetic sculpture. Through the use of internal stops, each arm is constrained to rotate a maximum of 68.75° (1/2 the Golden Angle) relative to its neighboring layer. See this sculpture in motion at http://www.johnedmark.com/#/rotating/.

doesn't do what I want it to do, and mostly that's not a happy thing. But once in a very blue moon, when it doesn't do what I wanted [it] to do, it actually does something *better*. With the lollipopter, for instance, I was determined to try to make something that would transform between being flat and being pinecone-like. I wanted to make something with arms that I could turn so that, first, the top arm would turn, and then each subsequent arm would turn the one beneath it until it became this volumetric pinecone-like thing. That was my plan. But when I made the first prototype, and started turning it, I found that it was getting harder and harder to turn with each revolution, and beyond a certain point, I couldn't get it to turn at all. I was so disappointed! So there I was, very annoyed, very bummed out that my great plans had come to nought. But then, as I was sitting there, bemoaning my ill fate, I sort of absent-mindedly twisted the central shaft, and much to my surprise the arms rotated and separated from each other in sequence, automatically! I did not design that. I was trying to do something that was much more prosaic, while this much better idea was hiding there waiting to be discovered. I've sometimes wondered: had I happened to have made the original prototype in such a way that it didn't jam part way through turning it by hand, would I ever have stumbled upon this behavior?

I'm a big advocate for the importance of play in our lives. Play is essential. Play is how we learn. If I'm going to be creative and inventive, I have to be in a playful frame of mind. I have to feel free to be silly, to make a fool of myself, to not be worried about being judged. For me, play is inseparable from curiosity. The two of them go hand-in-hand. It's through curiosity that one doesn't just look at things and accept them as they are, but instead wonders: Why is it like that? How did it become that way? How does it work? What if *this* happened to it? Why couldn't *that* be done? These kinds of questions are all part of play and are part of the dialogue I have with my pieces, both while working on them and after I "complete" them.

Q: *It's so interesting to hear you say that because that is exactly how a lot of mathematicians would describe the research process.*

JE: That's what I often hear. And I think that's also how a lot of artists would describe their process. While I don't think of myself as a math artist, math has been a very powerful tool in my work. Trigonometry is not one of the things you hear people talking affectionately about, and I understand that because *I'm* not particularly fond of trigonometry. In high school trig class, I remember thinking, "Why are we doing this?"

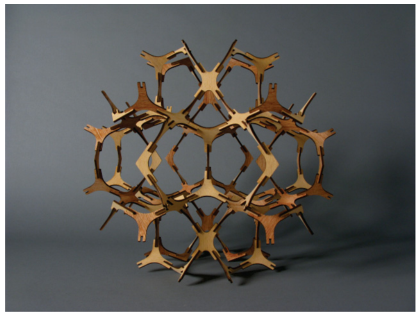

Figure 4 *Stella Octangula-Octahedron Space Packing*; laser-cut plywood, 2010. The stella octangula can be seen as either an octahedron with a tetrahedron placed on each face, or as the union of two overlapping tetrahedra. In combination with the octahedron the stella octangula can fill space. See this sculpture in motion at http://www.johnedmark.com/#/space-packings/.

For geometry, I didn't have to ask that question because I thought it was so fascinating to use just a compass and straightedge to prove things. That didn't need justification for me. Trigonometry did not have that appealing elegance. I didn't use it really for many years, but then I found myself trying to answer certain geometric questions, and I realized, "Oh! The answer to this is what trigonometry is designed to help me find!" I'm grateful that trigonometry is there and I use it, but it's never become something that's intuitive for me.

Q: *What are you thinking about these days?*

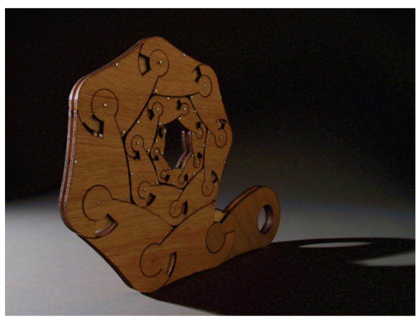

Figure 5 *Roll-Up Spiral*; laser-cut plywood, nails, string, 2002. This logarithmic spiral is furled and unfurled by pulling and releasing a string that passes through the inner radius of each section. To see this sculpture in motion, visit http://www.johnedmark.com/#/spirals/.

JE: I'm continuing to explore new Bloom designs. One direction I've been pursuing is developing what you might think of as overlapping behaviors, with elements traveling at different speeds, or in opposite directions. I have other areas I'm exploring as well, but the Blooms technology keeps drawing me back as it reveals new expressive potential.

PROOFS WITHOUT WORDS

$n^4 + 4^n$ is Composite for $n \geq 2$

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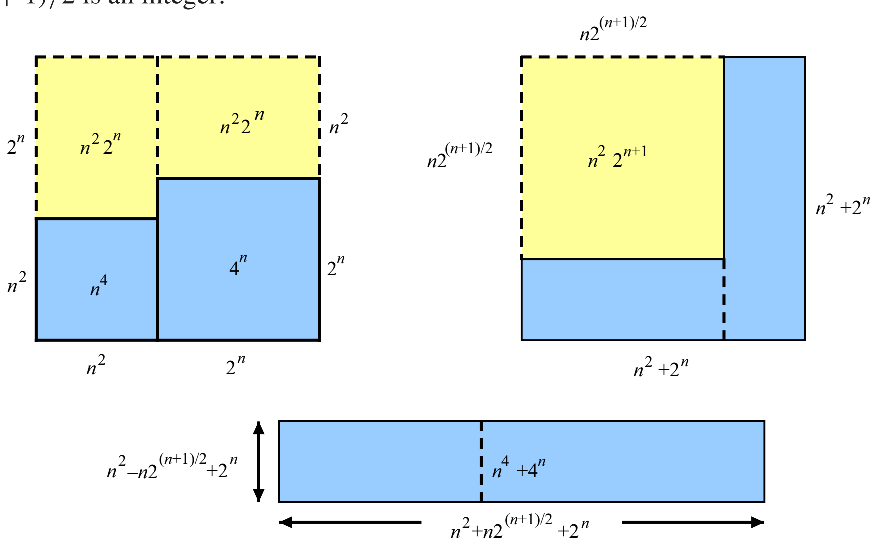
Since $n^4 + 4^n$ is even for n even, we need only consider n odd, in which case $(n+1)/2$ is an integer:



NOTE: Each factor is greater than 1 for $n \geq 3$ since completing the square yields

$$n^4 + 4^n = \left[\left(n + 2^{(n-1)/2}\right)^2 + 2^{n-1}\right]\left[\left(n - 2^{(n-1)/2}\right)^2 + 2^{n-1}\right].$$

EXAMPLE: $15^4 + 4^{15} = 1{,}073{,}792{,}449 = (36{,}833)(29{,}153)$.

Establishing this result was the subject of Problem 493 in the September 1962 issue of this MAGAZINE (p. 249). Solutions appeared in the March 1963 issue (pp. 138-139) and errata in the September 1963 issue (p. 265). See Nelsen [1, p. 35] for another factorization of a sum of two squares.

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[1] Nelsen, R. B. (2000). *Proofs Without Words II: More Exercises in Visual Thinking*. Washington D C : Mathematical Association of America.

Summary. We show that $n^4 + 4^n$ is composite for $n \geq 2$ by illustrating a factorization of $n^4 + 4^n$ for n odd (since $n^4 + 4^n$ is even for n even.)

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The Magic of Cyclic Polygons

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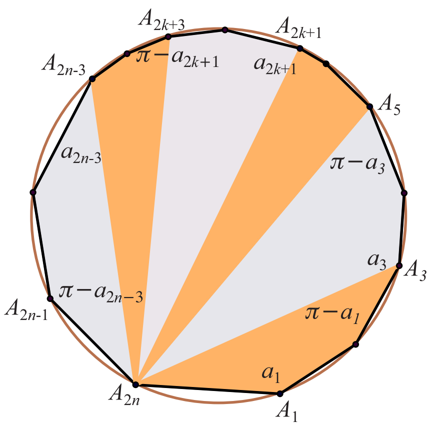
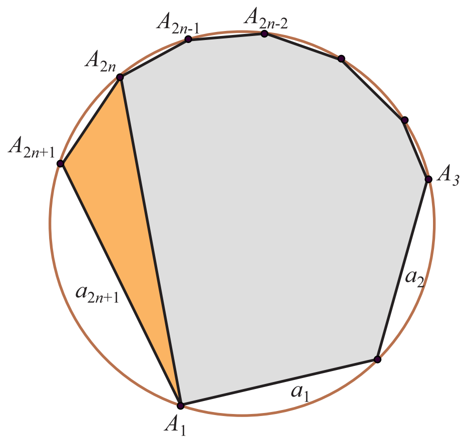
Both pairs of opposite angles of a cyclic quadrilateral sum to π. What can be said about alternating angle sums in other cyclic polygons?

Proposition 1. *The sum of any n alternating angles of a cyclic $2n$-gon is $(n-1)\pi$. The sum of any $n+1$ alternating angles of a cyclic $(2n+1)$-gon, increased by the angle subtended by the $(2n+1)$-st side, is $n\pi$.*

Proof.





∎

Summary. A cyclic $2n$-gon is divided into $n-1$ cyclic quadrilaterals to determine the sum of its alternating angles. A cyclic $(2n+1)$-gon divided into a cyclic $2n$-gon plus a triangle to determine the sum of its alternating angles, increased by an angle subtended by the $(2n+1)$-st side. Note that this includes the case of a triangle (for which $n = 1$).

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MSC: 51M04

PROBLEMS

Proposals

To be considered for publication, solutions should be received by July 1, 2021.

2111. *Proposed by Enrique Treviño, Lake Forest College, Lake Forest, IL.*

Evaluate

$$\sum_{n=0}^{\infty} \frac{\binom{4n}{2n}}{4^{2n}(2n+1)(2n+2)}.$$

2112. *Proposed by Souvik Dey (grad student), University of Kansas, Lawrence, KS.*

Let R be an integral domain and I and J be two ideals of R such that IJ is a non-zero principal ideal. Prove that I and J are finitely-generated ideals.

2113. *Proposed by George Stoica, Saint John, NB, Canada.*

Let A be an $n \times n$ complex matrix such that $\det(A^k + I_n) = 1$ for $k = 1, 2, \dots, 2^n - 1$.

(a) Prove that $A^n = O_n$.

(b) Show that the result does not hold if $2^n - 1$ is replaced by any smaller positive integer.

2114. *Proposed by Robert Haas, Cleveland Heights, OH.*

Find all configurations of four points in the plane (up to similarity) such that the set of distances between the points consists of exactly two lengths.

2115. *Proposed by H. A. ShahAli, Tehran, Iran.*

Let A and B be two distinct points on a circle and let k be a positive rational number.

Math. Mag. **94** (2021) 71–78. doi:10.1080/0025570X.2021.1850139

We invite readers to submit original problems appealing to students and teachers of advanced undergraduate mathematics. Proposals must always be accompanied by a solution and any relevant bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Proposals and solutions should be written in a style appropriate for this MAGAZINE.

Authors of proposals and solutions should send their contributions using the Magazine's submissions system hosted at `http://mathematicsmagazine.submittable.com`. *More detailed instructions are available there. We encourage submissions in PDF format, ideally accompanied by* LaTeX *source. General inquiries to the editors should be sent to* `mathmagproblems@maa.org`.

(a) Give a compass and straightedge construction of a point C on the circle such that $AC/BC = k$.

(b) Give a compass and straightedge construction of a point C on the circle such that $AC \cdot BC = k$. As part of your solution, find the restrictions on k in terms of AB and the radius of the circle necessary for such a C to exist.

Quickies

1107. *Proposed by Richard Stephens, Columbus State University (emeritus), Columbus, GA.*

For $n \in \mathbb{N}$, $k \in \mathbb{R}$, $k > n \geq 3$, we let x be the unique positive solution of the equation

$$1 + x + x^2 + \cdots + x^{n-1} = k. \tag{1}$$

Show that the following inequalities hold.

$$\delta_2 + \frac{1}{n} < x < \delta_2 \left(1 + \frac{2k\delta_2^2}{(n-2)(n-2+\delta_1) + 2\delta_2}\right),$$

where

$$\delta_1 = \sqrt{n^2 - 4n + 4k} \text{ and } \delta_2 = 1 - \frac{1}{k}.$$

1108. *Proposed by Ovidiu Furdui and Alina Sîntămărian, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.*

Given $a, b > 0$, calculate

$$\int_0^\infty \frac{x^{a-1}}{(x^a + 1)^2(x^b + 1)}\, dx.$$

Solutions

The largest divisor of $n^k - n$ **February 2020**

2086. *Proposed by David M. Bradley, University of Maine, Orono, ME.*

Let $f(k)$ denote the largest integer that is a divisor of $n^k - n$ for all integers n. For example, $f(2) = 2$ and $f(3) = 6$. Determine $f(k)$ for all integers $k > 1$.

Solution by the Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

To simplify notation, we write $g_k(n) = n^k - n$.

First, we prove two lemmas.

Lemma 1. *For every $k > 1$, $f(k)$ is square-free, i.e., if p is a prime then p^2 does not divide $f(k)$.*

Proof. Note that $g_k(p) = p(p^{k-1} - 1)$. If p^2 divided $g_k(p)$ then p would divide $p^{k-1} - 1$. But this would imply that p divides 1, giving a contradiction. ■

Lemma 2. *If p is a prime, then p divides $f(k)$ if and only if $p - 1$ divides $k - 1$.*

Proof. ($\Leftarrow$) If $k - 1 = (p - 1)\ell$ for some $\ell \geq 1$, then

$$g_k(n) = n((n^\ell)^{p-1} - 1).$$

If p divides n then it divides $g_k(n)$ too. If p does not divide n then by Fermat's little theorem p divides $(n^\ell)^{p-1} - 1$. Hence p divides $g_k(n)$ for every n, and this implies that p divides $f(k)$.

($\Rightarrow$) Assume a prime p divides $f(k)$. This means that p divides $g_k(n) = n(n^{k-1} - 1)$ for every n. Pick n to be a primitive root modulo p (which, by a well-known result in number theory, always exists). Then $1, n, n^2, \dots, n^{p-2}$, are distinct modulo p. Since p does not divide n, it must divide $n^{k-1} - 1$. Using the Euclidean algorithm we write $k - 1 = (p - 1)\ell + i$, with $\ell \geq 0$, $0 \leq i < p - 1$. By Fermat's little theorem $n^{p-1} \equiv 1 \pmod{p}$, hence

$$n^{k-1} = n^{(p-1)\ell+i} \equiv n^i \pmod{p}.$$

Since p divides $n^{k-1} - 1$ we have $n^{k-1} \equiv 1 \pmod{p}$, hence $n^i \equiv 1 \pmod{p}$. Since $1, n, \dots, n^{p-2}$ are distinct modulo p, we must have $i = 0$. Therefore $k - 1 = (p - 1)\ell$, i.e., $p - 1$ divides $k - 1$. ■

Lemmas 1 and 2 allow us to determine $f(k)$:

$$f(k) = \prod_{\substack{d|k-1 \\ d+1 \text{ is prime}}} (d + 1).$$

Example: To compute $f(19)$ we find the divisors of $19 - 1 = 18 : 1, 2, 3, 6, 9, 18$, add 1 to each of them: $2, 3, 4, 7, 10, 19$, then multiply the primes appearing on this list: $2 \cdot 3 \cdot 7 \cdot 19 = 798$. Thus $f(19) = 798$.

Editor's Note. It is immediate that $f(2j) = 2$. The proposer points out that by the von Staudt–Clausen theorem, $f(2j + 1)$ is the denominator of B_{2j}, the $2j$th Bernoulli number.

Also solved by Elijah Bland & Brooke Mullins, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), Robert Calcaterra, William Chang, John Christopher, Prithwijit De & B. Sury (India), Joseph DiMuro, Dmitry Fleischman, George Washington University Problems Group, Justin Haverlick, Omran Kouba (Syria), Sushanth Satish Kumar, Elias Lampakis (Greece), László Lipták, José Heber Nieto (Venezuela), Joel Schlosberg, Randy K. Schwartz, Doga Can Sertbas (Turkey), Jacob Siehler, John H. Smith, Albert Stadler (Switzerland), David Stone & John Hawkins, Edward White & Roberta White, and the proposer. There was one incomplete or incorrect solution.

A limit involving a recursively defined sequence **February 2020**

2087. *Proposed by Florin Stanescu, Şerban Cioiculescu School, Găeşti, Romania.*

Consider the sequence defined by $x_1 = a > 0$ and

$$x_n = \ln\left(1 + \frac{x_1 + x_2 + \cdots + x_{n-1}}{n - 1}\right) \quad \text{for } n \geq 2.$$

Compute $\lim_{n\to\infty} x_n \ln n$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is 2. A simple induction argument shows that $x_n > 0$ for all $n \geq 1$. Now, let $S_n = x_1 + x_2 + \cdots + x_n$ and define $\sigma_n = S_n/n$. Using the well-known inequality $\ln(1+x) \leq x$ which is valid for $x > -1$ (with equality if and only if $x = 0$), we conclude that

$$S_n - S_{n-1} = x_n = \ln\left(1 + \frac{S_{n-1}}{n-1}\right) \leq \frac{S_{n-1}}{n-1}$$

or equivalently $\sigma_n \leq \sigma_{n-1}$ for $n \geq 2$. So, the sequence $(\sigma_n)_{n\geq 1}$ is positive and decreasing, and since $x_n = \ln(1+\sigma_{n-1})$ the sequence $(x_n)_{n\geq 1}$ is also positive decreasing. Let $\ell = \lim_{n\to\infty} x_n$. By Cezáro's lemma we know that $\ell = \lim_{n\to\infty} \sigma_n$ and the equality $x_n = \ln(1+\sigma_{n-1})$ implies that $\ell = \ln(1+\ell)$, and consequently $\ell = 0$.

Now, because

$$\lim_{x\to 0} \ln(1+x)/x = 1$$

we conclude that $\lim_{n\to\infty} x_n/\sigma_{n-1} = 1$ On the other hand

$$\sigma_n = \sigma_{n-1} - \frac{1}{n}\left(\sigma_{n-1} - x_n\right) = \sigma_{n-1} - \frac{\sigma_{n-1} - \ln(1+\sigma_{n-1})}{n}.$$

But $\ln(1+x) = x - (1/2)x^2 + O(x^3)$ for small x, so

$$\sigma_n = \sigma_{n-1} - \frac{1}{2n}\sigma_{n-1}^2 + O\left(\frac{\sigma_{n-1}^3}{n}\right).$$

In particular, σ_n, σ_{n-1}, x_n, and x_{n+1} are all equivalent as $n \to \infty$. Now

$$1 + \sigma_n = (1+\sigma_{n-1})\left(1 - \frac{1}{2n}\sigma_{n-1}^2 + O\left(\frac{\sigma_{n-1}^3}{n}\right)\right).$$

So

$$x_{n+1} = x_n + \ln\left(1 - \frac{1}{2n}\sigma_{n-1}^2 + O\left(\frac{\sigma_{n-1}^3}{n}\right)\right) = x_n - \frac{1}{2n}\sigma_{n-1}^2 + O\left(\frac{\sigma_{n-1}^3}{n}\right).$$

Hence

$$n\left(\frac{1}{x_{n+1}} - \frac{1}{x_n}\right) = \frac{1}{2}\frac{\sigma_{n-1}^2}{x_n x_{n+1}} + O\left(\sigma_{n-1}\right).$$

Thus

$$\lim_{n\to\infty} n\left(\frac{1}{x_{n+1}} - \frac{1}{x_n}\right) = \frac{1}{2}.$$

Consequently, the Stolz–Cesáro theorem implies that

$$\lim_{n\to\infty} \frac{1}{H_n}\cdot\frac{1}{x_n} = \frac{1}{2},$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the nth harmonic number. Finally, recalling that $H_n = \ln n + O(1)$ we conclude that $\lim_{n\to\infty} x_n \ln n = 2$, as claimed.

Also solved by Robert A. Agnew, Brian Bradie, Robert Calcaterra, Hongwei Chen, Kee-Wai Lau (Hong Kong), Albert Stadler (Switzerland), and the proposer.

A Fibonacci sum **February 2020**

2088. *Proposed by Mircea Merca, University of Craiova, Romania.*

Let n and t be nonnegative integers. Prove that

$$\sum_{k=0}^{2n}(-1)^k F_{tk} F_{2tn-tk} = -\frac{F_t}{L_t} F_{2tn},$$

where F_i denotes the ith Fibonacci number and L_i denotes the ith Lucas number.

Solution by G. C. Greubel, Newport News, VA.
More generally let

$$\phi_n = \frac{\mu^n - \nu^n}{\mu - \nu} \qquad \text{and} \qquad \theta_n = \mu^n + \nu^n,$$

where $\mu + \nu = a$ and $\mu\,\nu = -b$. Note that when $a = b = 1$, $\phi_n = F_n$ and $\theta_n = L_n$ by the Binet formulas.

We have

$$(\mu - \nu)^2 \phi_{tk} \phi_{t(2n-k)} = \theta_{2tn} - \mu^{2tn} \left(\frac{\nu}{\mu}\right)^{tk} - \nu^{2tn} \left(\frac{\mu}{\nu}\right)^{tk}.$$

Using the sums

$$\sum_{k=0}^{2n}(-1)^k = 1$$

$$\sum_{k=0}^{2n}(-1)^k \left(\frac{\nu}{\mu}\right)^{tk} = \frac{\mu^t}{\theta_t} \left(1 + \left(\frac{\nu}{\mu}\right)^{t(2n+1)}\right)$$

$$\sum_{k=0}^{2n}(-1)^k \left(\frac{\mu}{\nu}\right)^{tk} = \frac{\nu^t}{\theta_t} \left(1 + \left(\frac{\mu}{\nu}\right)^{t(2n+1)}\right)$$

we find that

$$\sum_{k=0}^{2n}(-1)^k \phi_{tk}\, \phi_{t(2n-k)} = \frac{1}{(\mu - \nu)^2} \left(\theta_{2tn} - \frac{2\,\theta_{t(2n+1)}}{\theta_t}\right)$$

$$= -\frac{1}{\theta_t} \frac{1}{(\mu - \nu)^2} \left(2\,\theta_{t(2n+1)} - \theta_t\, \theta_{2tn}\right) = -\frac{\phi_t}{\theta_t} \phi_{2tn}.$$

Letting $a = b = 1$ gives the desired result.

A similar argument shows that

$$\sum_{k=0}^{2n} \phi_{tk}\, \phi_{t(2n-k)} = \frac{(2\,n\,\phi_t\,\theta_{2tn} - \theta_t\,\phi_{2tn})}{(a^2 + 4\,b)\,\phi_t}$$

and hence

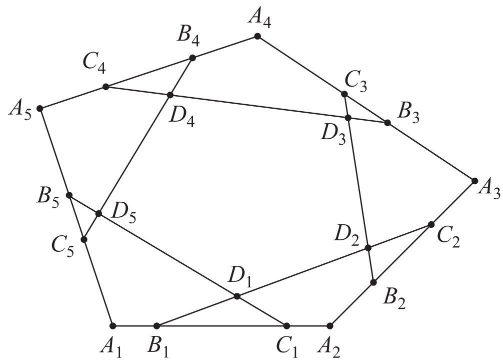
$$\sum_{k=0}^{2n} F_{tk}\, F_{t(2n-k)} = \frac{(2\,n\,F_t\,L_{2tn} - L_t\,F_{2tn})}{5\,F_t}.$$

Also solved by Michel Bataille (France), Brian Bradie, Robert Calcaterra, Dmitry Fleishman, Harris Kwong, Abhisar Mittal, José Heber Nieto (Venezuela), Angel Plaza (Spain), Albert Stadler (Switzerland), Michael Vowe (Switzerland), and the proposer.

A product of ratios for nested polygons **February 2020**

2089. *Proposed by Rick Mabry, LSU Shreveport, Shreveport, LA.*

Let $A_1, A_2, \ldots, A_n$ be the vertices of a convex n-gon in the plane. Identifying the indices modulo n, define the following points: Let B_i and C_i be vertices on $\overline{A_iA_{i+1}}$ such that $A_iB_i = C_iA_{i+1} < A_iA_{i+1}/2$ and let D_i be the intersection of $\overline{B_{i-1}C_i}$ and $\overline{B_iC_{i+1}}$. Prove that $\prod_{i=1}^{n}(B_iD_i)/(D_iC_i) = 1$.



Solution by José Heber Nieto, Universidad del Zulia, Maracaibo, Venezuela.
Let $\beta_i = \angle C_iB_iD_i$ and $\gamma_i = \angle D_iC_iB_i$. Applying the law of sines to triamgles $\triangle B_iC_iD_i$ and $\triangle B_{i-1}A_iC_i$ leads to

$$\frac{B_iD_i}{D_iC_i} = \frac{\sin\gamma_i}{\sin\beta_i} \quad \text{and} \quad \frac{B_{i-1}A_i}{A_iC_i} = \frac{\sin\gamma_i}{\sin\beta_{i-1}}.$$

Also, $A_iB_i = C_iA_{i+1}$ implies that $A_iC_i = B_iA_{i+1}$. Using these equations, we obtain

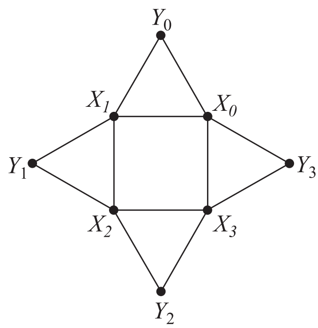
$$\begin{aligned}
\prod_{i=1}^{n}\frac{B_iD_i}{D_iC_i} &= \prod_{i=1}^{n}\frac{\sin\gamma_i}{\sin\beta_i} = \frac{\prod_{i=1}^{n}\sin\gamma_i}{\prod_{i=1}^{n}\sin\beta_i} = \frac{\prod_{i=1}^{n}\sin\gamma_i}{\prod_{i=1}^{n}\sin\beta_{i-1}}\\
&= \prod_{i=1}^{n}\frac{\sin\gamma_i}{\sin\beta_{i-1}} = \prod_{i=1}^{n}\frac{B_{i-1}A_i}{A_iC_i} = \frac{\prod_{i=1}^{n}B_{i-1}A_i}{\prod_{i=1}^{n}A_iC_i}\\
&= \frac{\prod_{i=1}^{n}B_{i-1}A_i}{\prod_{i=1}^{n}B_iA_{i+1}} = \frac{\prod_{i=1}^{n}B_{i-1}A_i}{\prod_{i=1}^{n}B_{i-1}A_i} = 1.
\end{aligned}$$

Also solved by Robert Calcaterra, William Chang, Elton Bojaxhiu (Germany) & Enkel Hysnelaj (Australia), George Washington University Problems Group, Joel Schlosberg, and the proposer.

Matchings in a certain family of graphs **February 2020**

2090. *Proposed by Gregory Dresden, Washington & Lee University, Lexington, VA.*

Recall that a *matching* of a graph is a set of edges that do not share any vertices. For example, C_4, the cyclic graph on four vertices (i.e., a square), has seven matchings: the empty set, single edges (four of these), or pairs of opposite edges (two of these).
Let G_n be the (undirected) graph with vertices x_i and y_i, $0 \leq i \leq n-1$, and edges $\{x_i, x_{i+1}\}$, $\{x_i, y_i\}$, and $\{y_i, x_{i+1}\}$, $0 \leq i \leq n-1$, where the indices are to be taken modulo n. For example, G_4 is shown below. Determine the number of matchings of G_n.



Solution by the George Washington University Problems Group, George Washington University, Washington, DC.
The answer is 3^n. To see this, let $S = \{-1, 0, 1\}^n$, a set whose cardinality is clearly 3^n. We show that there is a bijection ϕ from S to the set of matchings of G_n. Let $a = (a_1, \ldots, a_n)$ be an element of S. We define $\phi(a)$ as follows:

$$\{x_i, x_{i+1}\} \in \phi(a) \text{ if and only if } a_i = 1 \text{ and } a_{i+1} = -1,$$

$$\{x_i, y_i\} \in \phi(a) \text{ if and only if } a_i = 1 \text{ and } a_{i+1} \neq -1, \text{ and}$$

$$\{x_{i+1}, y_i\} \in \phi(a) \text{ if and only if } a_i \neq 1 \text{ and } a_{i+1} = -1.$$

We now check that $\phi(a)$ is indeed a matching. The edges incident to y_i are not both in $\phi(a)$, since $\{x_i, y_i\} \in \phi(a)$ requires $a_i = 1$ but $\{x_{i+1}, y_i\} \in \phi(a)$ requires $a_i \neq 1$. Also, among the four edges incident to x_i, at most one can be chosen for $\phi(a)$, since including $\{x_i, x_{i-1}\}$, $\{x_i, y_{i-1}\}$, $\{x_i, y_i\}$, and $\{x_i, x_{i+1}\}$ require, respectively, the four mutually exclusive conditions (1) $a_i = -1$ and $a_{i-1} = 1$, (2) $a_i = -1$ and $a_{i-1} \neq 1$, (3) $a_i = 1$ and $a_{i+1} \neq -1$, and (4) $a_i = 1$ and $a_{i+1} = -1$.

Given a matching M, there is a unique $a \in S$ so that M is $\phi(a)$. To see this, let $a_i = 1$ if M contains $\{x_i, x_{i+1}\}$ or $\{x_i, y_i\}$, let $a_i = -1$ if M contains $\{x_{i-1}, x_i\}$ or $\{x_i, y_{i-1}\}$, and let $a_i = 0$ if x_i is not the endpoint of any edge in M. This element $a \in S$ is the only element in $\phi^{-1}(M)$. Hence ϕ is bijective.

Also solved by Elton Bojaxhiu (Germany) and Enkel Hysnelaj (Australia), Robert Calcaterra, Jiakang Chen, Eddie Cheng; Serge Kruk; Li Li & László Lipták (jointly), José H. Nieto (Venezuela), Kishore Rajesh, Edward Schmeichel, John H. Smith, and the proposer. There was one incomplete or incorrect solution.

Answers

Solutions to the Quickies from page 72.

A1107. To prove the lower bound, consider $g(x) = x^n - 1 - k(x-1)$ for all $x \geq 0$. For $x \neq 1$, (1) is equivalent to $g(x) = 0$. Since $k > n$, the solution of (1) must satisfy $x > 1$. Now for $x \geq 0$, $g(x)$ has exactly one minimum value at

$$x_0 = \left(\frac{k}{n}\right)^{1/(n-1)} > 1.$$

Since $g(1) = 0$, the solution of (1) must be greater than x_0. Hence

$$k(x-1) = x^n - 1 > x_0^n - 1 = \left(\frac{k}{n}\right)^{n/(n-1)} - 1 > \frac{k}{n} - 1.$$

Therefore

$$x > 1 + \frac{k-n}{kn} = \delta_2 + \frac{1}{n}$$

as desired.

For the upper bound, we first use the AM-GM inequality to show that we have the strict inequality

$$k - 1 - x^{n-1} = x + x^2 + \cdots + x^{n-2} > (n-2)x^{\frac{1}{n-2}\sum_{i=1}^{n-2} i} = (n-2)x^{\frac{n-1}{2}}.$$

Next, we observe that this inequality is equivalent to $t^2 + (n-2)t - (k-1) < 0$, where $t = x^{\frac{n-1}{2}}$. This implies that

$$t \leq \frac{\sqrt{(n-2)^2 + 4k - 4} - (n-2)}{2} = \frac{2(k-1)}{\sqrt{n^2 - 4n + 4k} + n - 2}.$$

Since (1) is equivalent to $x^n - 1 = k(x-1)$ (if $x \neq 1$), we have

$$k(x-1) = xt^2 - 1 < x\frac{2(k-1)^2}{n^2 - 4n + 2 + 2k + (n-2)\sqrt{n^2 - 4n + 4k}} - 1.$$

Hence

$$\begin{aligned} x &< \left(1 - \frac{1}{k}\right)\left(1 + \frac{\frac{2(k-1)^2}{k}}{(n-2)(n-2+\delta_1) + \frac{2(k-1)}{k}}\right) \\ &= \delta_2\left(1 + \frac{2k\delta_2^2}{(n-2)(n-2+\delta_1) + 2\delta_2}\right), \end{aligned}$$

as we wished to show.

A1108. The integral equals $1/(2a)$. Making the substitution $x = 1/y$, we have

$$I = \int_0^\infty \frac{x^{a-1}}{(x^a+1)^2(x^b+1)}\,dx = \int_0^\infty \frac{y^{a-1+b}}{(y^a+1)^2(y^b+1)}\,dy.$$

It follows that

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \left(\frac{x^{a-1}}{(x^a+1)^2(x^b+1)} + \frac{x^{a-1+b}}{(x^a+1)^2(x^b+1)}\right)dx \\ &= \frac{1}{2}\int_0^\infty \frac{x^{a-1}}{(x^a+1)^2}\,dx = -\frac{1}{2a(x^a+1)}\Big|_0^\infty = \frac{1}{2a}, \end{aligned}$$

as claimed.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Wolfram, Stephen, The empirical metamathematics of Euclid and beyond, writings.stephenwolfram.com/2020/09/the-empirical-metamathematics-of-euclid-and-beyond/.

Instructors of mathematics tout the subject as consisting of logical deductions from axioms, definitions, and undefined primitive terms, but we don't always highlight the logical structure underlying the particular topics that we teach. When I taught calculus, since the textbook did not offer such a perspective, I would give students a handout showing the dependence of important theorems on one another and on basic concepts, such as the notion of limit. Stephen Wolfram's latest essay details two major favors that he has done for mathematicians. The first is that he uses his software to show dozens of directed graphs illustrating levels of interdependence of the 465 theorems in Euclid's *Elements* and their dependence on 131 definitions and 10 axioms (the five postulates plus the five common notions). For example, the last theorem in the *Elements*, that there are five Platonic solids, rests on all 10 axioms and 219 previous theorems. Wolfram notes that Donald Knuth pointed out to him a predecessor in Thomas Harriot (1560–1621), who had listed the full dependency table for the theorems in Book 1 of the *Elements*. The second favor is that Wolfram has added automated theorem proving to Mathematica and to the Wolfram Language. After formalization of the axioms for geometry, this tool can prove all of the theorems in the *Elements* and potentially discover new ones. Wolfram illustrates the power of this automated theorem proving with details from the simpler example of the axioms of basic logic. Further, he shows how concepts of "higher" mathematics, such as a topological space being Hausdorff, can be formalized.

Ornes, Stephen, How close are computers to automating mathematical reasoning?, quantamagazine.org/how-close-are-computers-to-automating-mathematical-reasoning-20200827/.

Could automated theorem provers "prove" useful to mathematicians in generating new mathematics? How about interactive theorem provers to check mathematician-made proofs? Should we look forward to such computerized assistants being used as referees by journals to check proofs in submitted manuscripts? Mathematicians' attitudes toward such "progress" are varied. Using such tools depends on formalizing the concepts involved and then coding the formalization in a particular language—difficult activities that some mathematicians have no interest whatever in undertaking, preferring instead to "do mathematics" the "old-fashioned" way.

Buyssse, Martin, Theorems of Euclidean geometry through calculus, arxiv:2007.00002v1.

Author Buyesse has us imagine that Newton was an early Greek. Equipped with calculus, Newton could have used infinitesimals to derive virtually all the classical theorems of Euclidean geometry and of trigonometry (you name the theorem, it's done here). Although differential equations deliver the formulas, the proofs remain geometric. "The main advantage of this method is that the theorem does not need to be known. We start with a function . . . and observe the way it behaves, at first order, under small deviations of some quantities." In effect, Buysse offers a mathematical analogue of how physicists derive formulas via dimensional analysis.

Math. Mag. **94** (2021) 79–80. doi:10.1080/0025570X.2021.1850138 © Mathematical Association of America

Marcolli, Matilde, *Lumen Naturae: Visions of the Abstract in Art and Mathematics*, MIT Press, 2020; xix + 369 pp, $44.95. ISBN 978-0-262-04390-8.

This is an extraordinary, fascinating, and astonishing book. Author Marcolli, a mathematician and theoretical physicist, writes about how contemporary art (particularly abstract art) explores concepts such as space, randomness, matter, and complexity. The book is intended for "scientifically minded people." It is extraordinary, because unlike most books that try to pair mathematics and art, Marcolli does not avoid delving into and explaining mathematics, such as vector spaces, the real projective plane, transformations of measure spaces, entropy, Shannon and quantum information, continuous probability distributions, the Einstein–Hilbert action functional for gravity, Apollonian circle packing, the Casimir effect, the Higgs field, the Möbus inversion formula, differential graded algebras, zeta functions—and more. Although much of that exposition only touches the surface, the pointers to analogues and representations in art make for fascinating reading. The breadth of the undertaking is astonishing, and it is furthered by the inclusion of almost 250 full-color reproductions of works of art illustrating the author's ideas. Each chapter ends with pages of references to further reading about the art and the artists and about the mathematics and science discussed.

Mackenzie, Dana, *What's Happening in the Mathematical Sciences*, *Vol. 11*, American Mathematical Society, 2019; vi + 136 pp, $25(P) ($22.50 to MAA members). ISBN 978-1-4704-4163-0.

This volume, the first in its series in four years, contains nine exciting and stimulating vignettes about discoveries and progress in mathematics, at a level that students will find understandable and inspirational. The topics are mathematical criteria for gerrymandering, metabolism and obesity, efficient commuting (bike-sharing and ride-sharing), colliding black holes, "expanders" (sparsely-connected graphs in which information can expand rapidly), algorithms for quantum computers, topological data analysis, the cap set conjecture and the game of Set®, and the asymptotic Fermat's last theorem for number fields.

Humphreys, Joe, What to do about George Berkeley, Trinity figurehead and slave owner?, irishtimes.com/culture/what-to-do-about-george-berkeley-trinity-figurehead-and-slave-owner.1.4277555.

Named after George Berkeley are a college library and annual medals at Trinity College Dublin, not to mention a city and a university in California. Not widely known is that he bought slaves and "justified slavery as a path to Christian conversion." Despite the opprobrium associated with that, the university named after him is more closely associated in the public mind with "a legacy of radical student and community politics." Of course, libraries and medals can be renamed, even cities (New Amsterdam, Constantinople, Christiania). But perhaps we shouldn't run the risk of naming cities, buildings, streets, etc. after people, much less erecting statues in their honor; after all, numbered and lettered streets could suffice, and New York City even numbers its public schools. Nevertheless, those who *deliberately* forget and neglect history . . . .

Ball, Laura, Why mathematicians should stop naming things after each other: A past generation's glory can be the next generation's headache, nautil.us/issue/89/the-dark-side/why-mathematicians-should-stop-naming-things-after-each-other.

Consider "A Calabi-Yau manifold is a compact, complex Kähler manifold with a trivial first Chern class." Author Ball points out that the name of the mathematician to whom a concept or theorem is attributed conveys no intrinsic information about it and contributes to obfuscation. And what to do if the mathematician made remarks or held views later deemed objectionable?

Holly, Jan E., and David Krumm, Morikawa's unsolved problem, arxiv: 2008.00922v1.

The authors "solve" the last unsolved problem in the Japanese *sangaku*, geometric problems hung on tablets in temples from the 17th century on. The problem asks for an algebraic formula for the length of the side of a square inscribed between two circles; the authors show that there is no such formula.